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Research Article

Some Fixed-Point Theorems over a Generalized $\mathscr{F}$-Metric Space

Kushal Roy,[1] Hossein Alaeidizaji,[2] Mantu Saha,[1] Babak Mohammadi,[3] and Vahid Parvaneh [4]

[1]*Department of Mathematics, The University of Burdwan, Purba Bardhaman, 713104 Bardhaman, West Bengal, India*
[2]*Department of Mathematics, Payame Noor University, P.O. Box. 19395-3697, Tehran, Iran*
[3]*Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran*
[4]*Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Kermanshah, Iran*

Correspondence should be addressed to Vahid Parvaneh; zam.dalahoo@gmail.com

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In this article, the concept of sequential $\mathscr{F}$-metric spaces has been introduced as a generalization of usual metric spaces, b-metric spaces, JS-metric spaces, and mainly $\mathscr{F}$-metric spaces. Some topological properties of such spaces have been discussed here. By considering this notion, we prove fixed-point theorems for some classes of contractive mappings over such spaces. Examples have been given in order to examine the validity of the underlying space and in support of our fixed-point theorems. Moreover, our fixed-point theorem is applied to obtain solution of a system of linear algebraic equations.

1. Introduction and Preliminaries

Two distance-controlled functions have been used extensively by the researchers working on fixed-point theory for obtaining fixed points of mappings such as contractive or expansive mappings in nature. Also, the polygonal inequality involved in a metric-like structure plays vital role for defining the topology on such space. But nowadays, after the introduction of JS-metric space, the latest fashion is to define a metric-type space which does not involve any type of polygonal inequality (see [1–3]). There is an immense literature in fixed-point theory and applications. For instance, in [4], a class of generalized (ψ, α, β)-weak contraction has been introduced, and some fixed-point theorems in the framework of partially ordered metric spaces have been proved. The authors also applied their results to a first-order ordinary differential equation.

Now, we remember some efforts on F-metric spaces.

In [5], Asif and Nazam noticed that the existence of fixed points of F-contractions, in an F-metric space, can be ensured with restricted conditions on the Wardowski function $F : (0,\infty) \longrightarrow R$. They obtained some fixed-point results for both single and set-valued Reich-type F-contractions in F-metric spaces. To show the usability of our results, we present two examples. Also, an application to functional equations is presented.

In [6], Jahangir et al. investigated some properties of F-metric spaces. They presented a simple proof to show that the natural topology induced by an F-metric is metrizable. They also presented a method to construct F-metric spaces from bounded metric spaces. They also showed that F-metrics are not necessarily jointly continuous functions. They showed that the Nadler fixed-point theorem and, therefore, the Banach contraction principle in the framework of F-metric spaces, the Schauder fixed-point theorem in F-normed spaces, and also some related F-metric fixed-point results can be reduced to their original metric versions.

We now give some definitions of generalized metric-type spaces which are relevant to our research work.

Definition 1 (b-metric space) [7, 8]. Let Δ be a nonempty set and h be a real number satisfying $h \geq 1$. A function $\sigma_b : \Delta \times \Delta \longrightarrow \mathbb{R}^+$ is a $b-$metric on Δ provided that

(1) $\sigma_b(a, c) = 0$ if and only if $a = c$

(2) $\sigma_b(a,c) = \sigma_b(c,a)$ for all $a, c \in \Delta$

(3) $\sigma_b(a,c) \le h[\sigma_b(a,e) + \sigma_b(e,c)]$ for all $a, c, e \in \Delta$.

The space (Δ, σ_b) is called a b-metric space.

Let Δ be a nonempty set and $\sigma_g : \Delta \times \Delta \longrightarrow [0,\infty)$ be a mapping. For any $a \in \Delta$, let us define the set

$$C(\sigma_g, \Delta, a) = \left\{ \{a_n\} \subset \Delta : \lim_{n\longrightarrow\infty} \sigma_g(a_n, a) = 0 \right\}. \quad (1)$$

Definition 2 (JS-metric space) [9]. Let $\sigma_g : \Delta \times \Delta \longrightarrow [0,\infty)$ be a mapping such that

(1) $\sigma_g(a,c) = 0$ implies $a = c$

(2) for every $a, c \in \Delta$, we have $\sigma_g(a,c) = \sigma_g(c,a)$

(3) if $(a,c) \in \Delta \times \Delta$ and $\{a_n\} \in C(\sigma_g, \Delta, a)$, then $\sigma_g(a,c) \le l \limsup_{n\longrightarrow\infty} \sigma_g(a_n, c)$, for some $l > 0$.

The pair (Δ, σ_g) is called a generalized metric space, usually known as JS-metric space (JSMS).

Definition 3 ($\mathscr{F}$-metric space) [10]. Let Δ be a nonempty set. A mapping $\sigma_f : \Delta \times \Delta \longrightarrow [0,\infty)$ is said to be an $\mathscr{F}$-metric on Δ, if for all $a, c \in \Delta$, σ_f satisfies

(1) $\sigma_f(a,c) = 0$ if and only if $a = c$

$$\sigma_f(a,c) = \sigma_f(c,a) \quad (2)$$

(2) For every $m \in \mathbb{N}$ with $m \ge 2$, and for every $(u_i)_1^m$ with $(u_1, u_m) = (a,c)$, we have

$$\Omega(\sigma_f(a,c)) \le \Omega\left(\sum_{i=1}^{m-1} \sigma_f(u_i, u_{i+1})\right) + k, a \ne c, \quad (3)$$

where $\Omega : (0,\infty) \longrightarrow (-\infty,\infty)$ is an increasing function such that $\Omega(t_n) \longrightarrow -\infty$ for all 0-convergent sequence $\{t_n\}$ and $k \in [0,\infty)$.

The pair (Δ, σ_f) is called an $\mathscr{F}$-metric space.

Motivated from the previous definitions and based on these ideas, we now define a new generalized metric-type space in our next section.

2. Sequential $\mathscr{F}$-Metric Spaces

In this section, we introduce the concept of sequential $\mathscr{F}$-metric space. To develop such a notion, first we define $S(\sigma, \Delta, a) := \{\{a_n\} \subset \Delta : \lim_{n\longrightarrow\infty} \sigma(a_n, a) = 0\}$, where $\sigma : \Delta \times \Delta \longrightarrow [0,\infty)$ is a given mapping.

Definition 4. Let Δ be a nonempty set. A mapping $\sigma : \Delta \times \Delta \longrightarrow [0,\infty)$ is said to be a sequential $\mathscr{F}$-metric if for all $a, b \in \Delta$

$(F1)$ $\sigma(a,b) = 0$ implies $a = b$

$(F2)$ $\sigma(a,b) = \sigma(b,a)$

$(F3)$ $\Omega(\sigma(a,b)) \le \Omega(\limsup_{n\longrightarrow\infty} \sigma(a_n, b)) + p$, for all $\{a_n\} \in S(\sigma, \Delta, a)$, where $\Omega : [0,\infty] \longrightarrow [-\infty,\infty]$ is an increasing function with $\Omega(t) = \infty$ iff $t = \infty$, $\Omega(t_n) \longrightarrow -\infty$ for all 0-convergent sequence $\{t_n\}$ and $p \ge 0$.

The triplet (Δ, σ, Ω) is called a sequential $\mathscr{F}$-metric space (SFMS). A SFMS indicated simply as (Δ, σ).

Example 5. Let $\Lambda = N$ and the metric $\sigma : \Lambda^2 \longrightarrow [0,\infty)$ be defined by

$$\begin{cases} \sigma(1,1) = 0; & \\ \sigma(n,n) = e - 1, & \text{for } n \ge 2; \\ \sigma(1,n) = \sigma(n,1) = \cosh\left(\dfrac{1}{n+1}\right) - 1, & \text{for } n \ge 2; \\ \sigma(n,m) = \sigma(m,n) = \cosh(mn) - 1, & \text{for all } n, m \ge 2 \text{ with } n \ne m. \end{cases} \quad (4)$$

Also, let $\Omega(x) = 1 - 1/\sqrt{x}$.

For $n \ge 2$, $S(\sigma, \Lambda, n) = \varnothing$. Let $\{n_k\} \in S(\sigma, \Lambda, 1)$. If all but finitely many terms of $\{n_k\}$ are 1, then we have nothing to prove. So, suppose that $\{n_k\}$ only have finitely many 1's. Without loss of generality, we can exclude such 1's, and then, we get $\limsup_{k\longrightarrow\infty} \sigma(m, n_k) = \lim_{k\longrightarrow\infty} [\cosh(mn_k) - 1] = \infty$. Therefore, $1 - 1/\sqrt{\sigma(1,m)} \le 1 - 1/\sqrt{\limsup_{k\longrightarrow\infty} \sigma(n_k, m)}$ for all $m \ge 2$.

Hence, σ is a sequential p-metric on Λ for $\Omega(x) = 1 - 1/\sqrt{x}$ for all $x > 0$ and $p = 1$.

Note that taking $a = 3$, $b = 2$, and $c = 1$, we see that $\sigma(a,b) - (\sigma(a,c) - \sigma(c,b)) = \cosh 6 - 1 - (\cosh(1/3 - 1) + \cosh(1/4 - 1)) = 198.190377258$. So, σ is not a usual metric. Now, if a, b are sufficiently large and $c = 1$, again, the left-hand side in triangular inequality in a b-metric space is greater than the right-hand side. So, σ is not also a b-metric.

To show that σ is not an F-metric space, it is sufficient to take $m = 3$, a, b are sufficiently large and $c = 1$. So

$$\begin{aligned} \Omega(\sigma_f(a,c)) &= \Omega(\cosh(mn) - 1) \longrightarrow \infty, \\ \Omega\left(\sum_{i=1}^{2} \sigma_f(u_i, u_{i+1})\right) + k &= \Omega\left(\cosh\left(\frac{1}{n+1}\right) - 1 \right. \\ &\quad \left. + \cosh\left(\frac{1}{n+1}\right) - 1\right) + k \longrightarrow k. \end{aligned} \quad (5)$$

Proposition 6. *Any $\mathscr{F}$-metric space (Δ, D) is a SFMS.*

Proof. Since (Δ, D) is an $\mathscr{F}$-metric space, then for every $(a, b) \in \Delta^2$, for every $m \in \mathbb{N}$ with $m \geq 2$ and for every $(u_i)_1^m$ with $(u_1, u_m) = (a, b)$, we have

$$f(D(a,b)) \leq f\left(\sum_{i=1}^{m-1} D(u_i, u_{i+1})\right) + k, a \neq b \,; \text{where } (f,k) \in \mathscr{F} \times [0,\infty). \tag{6}$$

Therefore, it follows that

$$f(D(a,b)) \leq f(D(a,c) + D(c,b))) + k, \text{for } a \neq b \text{ and for any } c \in \Delta. \tag{7}$$

Thus, for any $a, b \in \Delta$, if we take $\{a_n\} \in S(\sigma, \Delta, a)$, then we see that

$$f^*(D(a,b)) \leq f^*(D(a,a_n) + D(a_n,b))) + k \text{ for all } n \geq 1, \tag{8}$$

where $f^* : [0,\infty] \longrightarrow [-\infty,\infty]$ is defined by $f^*(0) = -\infty$, $f^*(\infty) = \infty$, and $f^*(t) = f(t)$ for all $0 < t < \infty$. So

$$f^*(D(a,b)) \leq f^*\left(\limsup_{n\longrightarrow\infty} D(a_n, b)\right)\Bigg) + k, \tag{9}$$

and therefore, D also satisfies condition ($\mathscr{F}3$). Hence, D is a sequential $\mathscr{F}$-metric on Δ for the mapping f^* and $k \in [0,\infty)$.

Proposition 7. *Any JS-metric space $(\Delta, \bar{d})$ is a SFMS.*

Proof. Since $\bar{d}$ is a JS-metric, then there exists $l > 0$ such that for all $a, b \in \Delta$

$$\bar{d}(a,b) \leq l \limsup_{n\longrightarrow\infty} \bar{d}(a_n, b) \text{ for any } \{a_n\} \in S\left(\bar{d}, \Delta, a\right). \tag{10}$$

Thus

$$\Omega\left(\bar{d}(a,b)\right) \leq \Omega\left(\limsup_{n\longrightarrow\infty} \bar{d}(a_n, b)\right) + \log(l) \text{ for any } \{a_n\} \in S\left(\bar{d}, \Delta, a\right), \tag{11}$$

for all $a, b \in \Delta$. So, $\bar{d}$ also satisfies the third condition of Definition 4 for the function $\Omega(t) = \log t$ for all $0 < t < \infty$ and $\Omega(0) = -\infty$, $\Omega(\infty) = \infty$, and $p = \log(l)$. Hence, $\bar{d}$ is a sequential $\mathscr{F}$-metric on Δ.

Remark 8.

(i) A metric space, b -metric space [7, 8], metric-like space [11], and modular metric space with the Fatou property [12] are JS-metric spaces. Therefore, these spaces are also SFMS

(ii) Any $s-\text{relaxed}_p$ metric space [13] is an $\mathscr{F}$-metric space and therefore is also a SFMS. There exist b-metric spaces which are not $\mathscr{F}$-metric spaces (see [10]); therefore, our SFMS is a stronger concept than the concept of $\mathscr{F}$-metric space.

The following is an example of a SFMS which is not an $\mathscr{F}$-metric space as well as not a b-metric space.

Example 9. Let $\Delta = \mathbb{N}$ and $\sigma : \Delta^2 \longrightarrow [0,\infty)$ be defined by $\sigma(1,1) = 0$, $\sigma(n,n) = 2$, $\sigma(1,n) = \sigma(n,1) = 1/n^2$ for all $n \geq 2$, and $\sigma(n,m) = \sigma(m,n) = 1 + (1/m + n)$ for all $m, n \in \mathbb{N} \setminus \{1\}$ with $m \neq n$. Then for all $n \geq 2$, $S(\sigma, \Delta, n) = \varnothing$. Let $\{n_k\} \in S(\sigma, \Delta, 1)$. If all but finitely many terms of $\{n_k\}$ are 1, then we are done. So, let $\{n_k\}$ only have finitely many 1's. Without loss of generality, we can ignore such 1's, and then, we get $\limsup_{k\longrightarrow\infty} \sigma(m, n_k) = \lim_{k\longrightarrow\infty} [1 + (1/m + n_k)] = 1$. Therefore, $\sigma(1, m) \leq \limsup_{k\longrightarrow\infty} \sigma(m, n_k)$ for all $m \geq 2$. So, (Δ, σ) is a SFMS for $\Omega(t) = \log t$ for all $0 < t < \infty$, $\Omega(0) = -\infty$, $\Omega(\infty) = \infty$ and $p = 0$. If it is an $\mathscr{F}$-metric space, then there exists $(f, k) \in \mathscr{F} \times [0,\infty)$ such that

$$f(\sigma(n,m)) \leq f(\sigma(n,1) + \sigma(1,m)) + k \text{ for all } n, m(n \neq m) \geq 2. \tag{12}$$

Taking $n, m \longrightarrow \infty$, we see that $f(1 + (1/n + m)) \longrightarrow -\infty$, a contradiction. Hence, σ is not an $\mathscr{F}$-metric. In a similar way, we can show that σ is not a b-metric on Δ.

Definition 10. Let (Δ, σ) be a SFMS. Also let $\{a_n\}$ be a sequence in Δ and $a \in \Delta$.

(i) if $\{a_n\} \in S(\sigma, \Delta, a)$, $\{a_n\}$ is called convergent and converges to a

(ii) if $\lim_{n,m\longrightarrow\infty} \sigma(a_n, a_m) = 0$,$\{a_n\}$ is called Cauchy

(iii) if any Cauchy sequence in Δ is convergent, Δ is called complete.

Definition 11. Let (Δ, σ) and $(\mathscr{Y}, \sigma^*)$ be two SFMS. If for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $u \in \Delta$, $\sigma^*(Su, Sa) < \varepsilon$ whenever $\sigma(u, a) < \delta_\varepsilon$, then $S : \Delta \longrightarrow \mathscr{Y}$ is called continuous at a point $a \in \Delta$. S is said to be continuous on Δ if S is continuous at each point of Δ.

Proposition 12. *Let (Δ, σ) be a SFMS and $\{a_n\}$ be a convergent sequence converging to some $a, b \in \Delta$; then, $a = b$.*

Proof. If possible, let $a \neq b$. Then

$$\Omega(\sigma(a,b)) \leq \Omega\left(\limsup_{n\longrightarrow\infty} \sigma(a_n, b)\right) + p \longrightarrow -\infty, \tag{13}$$

since $\lim_{n\longrightarrow\infty} \sigma(a_n, b) = 0$, a contradiction. Hence, the result.

Proposition 13. *Let (Δ, σ) be a SFMS. If $\{a_n\}$ converges to some $a \in \Delta$, then $\sigma(a, a) = 0$.*

Proof. From the condition ($\mathscr{F}3$) of Definition 4, we have

$$\Omega(\sigma(a,a)) \leq \Omega\left(\limsup_{n\longrightarrow\infty}\sigma(a_n,a)\right) + p \longrightarrow -\infty, \tag{14}$$

implying that $\sigma(a,a)=0$.

Proposition 14. *Let $\{a_n\}$ be a Cauchy sequence in a SFMS (Δ,σ). If $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ which converges to $a\in\Delta$, then $\{a_n\}$ also converges to $a\in\Delta$.*

Proof. From condition ($\mathscr{F}3$) of Definition 4, we have

$$\Omega(\sigma(a_m,a)) \leq \Omega\left(\limsup_{k\longrightarrow\infty}\sigma\left(a_m,a_{n_k}\right)\right) + p, \text{for any } m\geq 1. \tag{15}$$

Taking $m\longrightarrow\infty$, we get $\Omega(\sigma(a_m,a))\longrightarrow-\infty$ as $m\longrightarrow\infty$. Therefore, $\sigma(a_m,a)\longrightarrow 0$ as $m\longrightarrow\infty$, that is, $a_m\longrightarrow a$ as $m\longrightarrow\infty$.

Remark 15. Here is an example of *JS*-metric space which is given by Senapati et al. [14]. Let $\Delta=\mathbb{R}^+\cup\{0,\infty\}$ and $\bar{d}:\Delta^2\longrightarrow[0,\infty)$ be defined by

$$\bar{d}(a,b) = (a+b, \text{either } a=0 \text{ or } b=0\,;\, 1+a+b, \text{otherwise}. \tag{16}$$

(i) In this space, we see that the sequence $\{1/n\}$ converges to 0, but it is not a Cauchy sequence. Since any *JS*-metric space is a SFMS also, therefore, in a SFMS, a convergent sequence is not necessarily Cauchy

(ii) Also, in a SFMS, if $\{a_n\}$ and $\{b_n\}$ are two sequences convergent to a and b, respectively, then $\{\sigma(a_n,b_n)\}$ may not be convergent to $\sigma(a,b)$. For this, let us consider two sequences $\{1/2n\}_{n\geq1}$ and $\{1/(2n+1)\}_{n\geq1}$, and then, both the sequences converge to 0 but $\bar{d}(1/2n,1/(2n+1))=1+1/2n+1/(2n+1)\longrightarrow 1\neq 0=\bar{d}(0,0)$.

Proposition 16. *In a SFMS (Δ,σ), if a self mapping T is continuous at $a\in\Delta$, then $\{a_n\}\in S(\sigma,\Delta,a)$ implies that $\{Ta_n\}\in S(\sigma,\Delta,Ta)$.*

Proof. Let $\varepsilon>0$ be given. Since T is continuous at a, then for any $\varepsilon>0$, there exists $\delta_\varepsilon>0$ such that $\sigma(u,a)<\delta_\varepsilon$, $u\in\Delta$, implies $\sigma(Tu,Ta)<\varepsilon$.

Let $\{a_n\}\in S(\sigma,\Delta,a)$. Since $\{a_n\}$ converges to a; for $\delta_\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $\sigma(a_n,a)<\delta_\varepsilon$ for all $n\geq N$. Therefore, for any $n\geq N$, $\sigma(Ta_n,Ta)<\varepsilon$, and thus, $Ta_n\longrightarrow Ta$ as $n\longrightarrow\infty$, that is, $\{Ta_n\}\in S(\sigma,\Delta,Ta)$.

Let (Δ,σ) be a SFMS with supporting function Ω and $p\geq 0$. Define

$$\begin{aligned} \mathrm{B}(a,\zeta) &:= \{b\in\Delta : \sigma(a,b)<\sigma(a,a)+\zeta\},\\ \mathrm{B}[a,\zeta] &:= \{b\in\Delta : \sigma(a,b)\leq\sigma(a,a)+\zeta\}, \end{aligned} \tag{17}$$

for all $a\in\Delta$ and $\zeta>0$.

Remark 17. The family

$$\tau_\sigma := \{\varnothing\}\cup\{\mathbb{U}(\neq\varnothing)\subset\Delta : \text{for any } a\in\mathbb{U}, \text{there exists } \zeta > 0 \text{ such that } \mathrm{B}(a,\zeta)\subset\mathbb{U}\} \tag{18}$$

forms a topology on Δ.

Definition 18. If there exists an open set $\mathbb{U}\subset\Delta$ such that $\mathbb{F}=\Delta\setminus\mathbb{U}$ in a SFMS (Δ,σ), then $\mathbb{F}$ is said to be closed.

Proposition 19. *Let (Δ,σ) be a SFMS and $\mathbb{F}\subset\Delta$ be closed. Let $\{a_n\}\subset\mathbb{F}$ such that $a_n\longrightarrow a$ as $n\longrightarrow\infty$. Then, $a\in\mathbb{F}$.*

Proposition 20. *Let (Δ,σ) be a complete SFMS and $\mathbb{F}\subset\Delta$ be closed. Then, the subspace $(\mathbb{F},\sigma)$ is also complete.*

Definition 21. In a SFMS (Δ,σ), for $\mathbb{A}\subset\Delta$, we define

$$\operatorname{diam}(\mathbb{A}) := \sup\{\sigma(a,b) : a,b\in\mathbb{A}\}. \tag{19}$$

Theorem 22. *Let (Δ,σ) be a complete SFMS and $\{\mathbb{F}_n\}$ be a decreasing sequence of nonempty closed subsets of Δ such that $\operatorname{diam}(\mathbb{F}_n)\longrightarrow 0$ as $n\longrightarrow\infty$. Then, $\cap_{n=1}^{\infty}\mathbb{F}_n=\{a\}$.*

3. Some Fixed-Point Theorems

Theorem 23 (Banach-type fixed-point theorem). *Let (Δ,σ) be a complete SFMS and $\varpi:\Delta\longrightarrow\Delta$ be a mapping which satisfies the following conditions:*

(i) $\sigma(\varpi(a),\varpi(b))\leq\mu\,\sigma(a,b)$ for all $a,b\in\Delta$ and for some $\mu\in(0,1)$

(ii) there exists $a_0\in\Delta$ such that $\delta(\sigma,\varpi,a_0):=\sup\{\sigma(\varpi^i a_0,\varpi^j a_0): i,j=1,2,\cdots\}<\infty$.

Then, ϖ has at least one fixed-point u in Δ. Moreover, if v and w are two fixed points of ϖ in Δ with $\sigma(v,w)<\infty$, then $v=w$.

Proof. Define $\delta(\sigma,\varpi^{p+1},a_0):=\sup\{\sigma(\varpi^{p+i}a_0,\varpi^{p+j}a_0): i,j\geq 1\}$ for every $p\geq 0$. Since $\delta(\sigma,\varpi,a_0)<\infty$, then $\delta(\sigma,\varpi^{p+1},a_0)<\infty$ for all $p\in\mathbb{N}\cup\{0\}$. Now

$$\sigma\left(\varpi^{p+i}a_0,\varpi^{p+j}a_0\right)\leq\mu\,\sigma\left(\varpi^{p-1+i}a_0,\varpi^{p-1+j}a_0\right)\leq\mu\,\delta(\sigma,\varpi^p,a_0), \tag{20}$$

for all $i,j\geq 1$ and $p\geq 1$.

Therefore, $\delta(\sigma, \varpi^{p+1}, a_0) \le \mu\delta(\sigma, \varpi^p, a_0)$ for all $p = 1, 2, 3, \cdots$, from which it follows that $\lim_{p \longrightarrow \infty}\delta(\sigma, \varpi^p, a_0) = 0$. Now, for any $1 \le n < m$, we have

$$\begin{aligned}\sigma(\varpi^n a_0, \varpi^m a_0) &= \sigma\left(\varpi^{n-1+1} a_0, \varpi^{n-1+(m-n+1)} a_0\right)\\ &\le \delta(\sigma, \varpi^n, a_0) \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned} \tag{21}$$

So, $\{\varpi^n a_0\}$ is Cauchy in Δ, and so, there exists some $u \in \Delta$ such that $\varpi^n a_0 \longrightarrow u$ as $n \longrightarrow \infty$. Thus, $\sigma(\varpi^{n+1} a_0, \varpi u) \le \mu\sigma(\varpi^n a_0, u) \longrightarrow 0$ as $n \longrightarrow \infty$. From Proposition 12, it follows that $\varpi u = u$ and u is a fixed point of ϖ.

Now, if v and w are two fixed points of ϖ in Δ with $\sigma(v, w) < \infty$, then we have $\sigma(v, w) = \sigma(\varpi v, \varpi w) \le \mu\sigma(v, w)$ which gives $\sigma(v, w) = 0$ implying that $v = w$.

Example 24. Let $\Delta = \mathbb{R}^+ \cup \{0\}$ endowed with the distance function:

$$\sigma(a, b) = \begin{cases} e^{a+b}, & \text{if } a \ne b\,;\\ 0, & \text{if } a = b.\end{cases} \tag{22}$$

Then, (Δ, σ) is a SFMS for $\Omega(t) = -1/t$ and $p = 1$. Now, let us define $\varpi : \Delta \longrightarrow \Delta$ as follows:

$$\varpi(a) = \begin{cases} 0, & \text{if } 0 \le a \le 1\,;\\ a - 1, & \text{if } a > 1.\end{cases} \tag{23}$$

Then, ϖ has all contractive conditions of Theorem 23 for $\mu = \exp(-1)$ and also satisfies all other additional conditions. Here, ϖ has a unique fixed-point 0 in Δ.

Theorem 25 (Reich-type fixed-point theorem). *Let (Δ, σ) be a complete SFMS and $\varpi : \Delta \longrightarrow \Delta$ satisfy*

(i) $\sigma(\varpi(a), \varpi(b)) \le \lambda\sigma(a, b) + \mu\sigma(a, \varpi(a)) + \nu\sigma(b, \varpi(b))$ for all $a, b \in \Delta$ and for $\lambda, \mu, \nu \in (0, 1)$ with $\lambda + \mu + \nu < 1$

(ii) there exists $a_0 \in \Delta$ such that $\delta(\sigma, \varpi, a_0) \coloneqq \sup\{\sigma(\varpi^i a_0, \varpi^j a_0)$: $i, j = 1, 2, \cdots\} < \infty$.

Then, the Picard iterating sequence $\{a_n\}$, $a_n = \varpi^n a_0$ for all $n \in \mathbb{N}$ converges to some $u \in \Delta$. If $\sigma(u, \varpi(u)) < \infty$ and $\Omega(\nu t) + p < \Omega(t)$ for all $t > 0$ then $u \in \Delta$ is a fixed point of ϖ. Moreover, if v is a fixed point of ϖ in Δ such that $\sigma(u, v) < \infty$ and $\sigma(v, v) < \infty$, then $u = v$.

Proof. Let us define $\delta(\sigma, \varpi^{p+1}, a_0) \coloneqq \sup\{\sigma(\varpi^{p+i} a_0, \varpi^{p+j} a_0)$: $i, j \ge 1\}$ for every $p \ge 0$. Since $\delta(\sigma, \varpi, a_0) < \infty$, then $\delta(\sigma, \varpi^{p+1}, a_0) < \infty$ for all $p \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned}&\sigma\left(\varpi^{p+i} a_0, \varpi^{p+j} a_0\right)\\ &\quad\le \lambda\sigma\left(\varpi^{p-1+i} a_0, \varpi^{p-1+j} a_0\right) + \mu\sigma\left(\varpi^{p-1+i} a_0, \varpi^{p+i} a_0\right)\\ &\qquad + \nu\sigma\left(\varpi^{p-1+j} a_0, \varpi^{p+j} a_0\right) \text{ for all } i, j \ge 1,\\ &\quad\le r\,\delta(\sigma, \varpi^p, a_0) \text{ for all } p \ge 1, \text{ where } r = \lambda + \mu + \nu < 1.\end{aligned} \tag{24}$$

Proceeding in a similar way as in Theorem 23, it follows that $\{\varpi^n a_0\}$, that is, $\{a_n\}$ is Cauchy in Δ, and so, there exists some $u \in \Delta$ such that $a_n \longrightarrow u$ as $n \longrightarrow \infty$. Therefore, we get

$$\begin{aligned}\sigma(a_{n+1}, \varpi u) &\le \lambda\sigma(a_n, u) + \mu\sigma(a_n, \varpi(a_n))\\ &\quad + \nu\sigma(u, \varpi u) \text{ for any } n \ge 1,\end{aligned} \tag{25}$$

which implies that $\limsup_{n \longrightarrow \infty}\sigma(a_{n+1}, \varpi u) \le \nu\sigma(u, \varpi u)$.

Thus, using condition $(\mathcal{F}3)$ of Definition 4, we have

$$\begin{aligned}\Omega(\sigma(u, \varpi u)) &\le \Omega\left(\limsup_{n \longrightarrow \infty}\sigma(a_{n+1}, \varpi u)\right) + p\\ &\le \Omega(\nu\sigma(u, \varpi u)) + p.\end{aligned} \tag{26}$$

Now, if $\sigma(u, \varpi u) > 0$, then by the assumed condition of the theorem, we see that $\Omega(\sigma(u, \varpi u)) \le \Omega(\nu\sigma(u, \varpi u)) + p < \Omega(\sigma(u, \varpi u))$, a contradiction. Thus, $\sigma(u, \varpi u) = 0$ gives $\varpi u = u$, and u is a fixed point of ϖ.

Now, if v is a fixed point of ϖ in Δ with $\sigma(u, v) < \infty$ and $\sigma(v, v) < \infty$, then we have $\sigma(u, v) = \sigma(\varpi u, \varpi v) \le \lambda\sigma(u, v) + \mu\sigma(u, \varpi u) + \nu\sigma(v, \varpi v) = \lambda\sigma(u, v)$, as $\sigma(v, v) = 0$, implying that $\sigma(u, v) = 0$, that is, $u = v$.

Corollary 26. *Let (Δ, σ) be a complete SFMS and $\varpi : \Delta \longrightarrow \Delta$ satisfies*

(i) $\sigma(\varpi(a), \varpi(b)) \le \nu[\sigma(a, \varpi(a)) + \sigma(b, \varpi(b))]$ for all $a, b \in \Delta$ and for $\nu \in (0, 1/2)$

(ii) there exists $a_0 \in \Delta$ such that $\delta(\sigma, \varpi, a_0) \coloneqq \sup\{\sigma(\varpi^i a_0, \varpi^j a_0)$: $i, j = 1, 2, \cdots\} < \infty$.

Then the Picard iterating sequence $\{a_n\}$, $a_n = \varpi^n a_0$ for all $n \in \mathbb{N}$ converges to some $u \in \Delta$. If $\sigma(u, \varpi(u)) < \infty$ and $\Omega(\nu t) + p < \Omega(t)$ for all $t > 0$, then $u \in \Delta$ is a fixed point of ϖ. Moreover, if v is a fixed point of ϖ in Δ such that $\sigma(u, v) < \infty$ and $\sigma(v, v) < \infty$, then $u = v$.

Proof. If we take $\lambda = 0$ and $\mu = \nu$, then this corollary follows from our Theorem 25.

Example 27. Let $\Delta = [0, 1]$ endowed with the distance function:

$$\sigma(a, b) = \begin{cases} |a - b| + |a - b|^2, & \text{if } a \ne b\,;\\ 0, & \text{if } a = b.\end{cases} \tag{27}$$

Then (Δ, σ) is a SFMS for $\Omega(t) = \log t$ and $p = \log 2$. Now, let $\varpi : \Delta \longrightarrow \Delta$ be defined as

$$\varpi(a) = \begin{cases} \dfrac{a}{4}, & \text{if } 0 \le a < \dfrac{1}{2}\,;\\[2mm] \dfrac{a}{5}, & \text{if } \dfrac{1}{2} \le a \le 1.\end{cases} \tag{28}$$

Then, ϖ satisfies the contractive condition of Corollary 26 for $\nu = 1/3$. Here, all other additional conditions are also satisfied. We see that ϖ has a unique fixed-point 0 in Δ.

Theorem 28 (Chatterjea-type fixed-point theorem). *Let (Δ, σ) be a complete SFMS and $\varpi : \Delta \longrightarrow \Delta$ be a mapping satisfying*

(i) $\sigma(\varpi(a), \varpi(b)) \leq \chi[\sigma(a, \varpi(b)) + \sigma(b, \varpi(a))]$ for all $a, b \in \Delta$ and for some $\chi \in (0, 1/2)$

(ii) there exists $a_0 \in \Delta$ such that $\delta(\sigma, \varpi, a_0) \coloneqq \sup \{\sigma(\varpi^i a_0, \varpi^j a_0): i, j = 1, 2, \cdots\} < \infty$.

Then the Picard iterating sequence $\{a_n\}$, $a_n = \varpi^n a_0$ for all $n \geq 1$ converges to some $u \in \Delta$. If $\lim \sup_{n\longrightarrow\infty}\sigma(a_n, \varpi(u)) < \infty$*, then $u \in \Delta$ is a fixed point of ϖ. Also, if v is a fixed point of ϖ in Δ such that $\sigma(u, v) < \infty$, then $u = v$.*

Proof. By similar argument as in Theorem 23, $\{a_n\}$ is a Cauchy sequence in Δ, and by the completeness of Δ, it converges to an element say $u \in \Delta$.

Now, for all $n \in \mathbb{N} \cup \{0\}$, $\sigma(a_{n+1}, \varpi u) = \sigma(\varpi a_n, \varpi u) \leq \chi[\sigma(a_n, \varpi u) + \sigma(a_{n+1}, u)]$, which implies that $\lim \sup_{n\longrightarrow\infty} \sigma(a_{n+1}, \varpi u) \leq \chi \lim \sup_{n\longrightarrow\infty}\sigma(a_n, \varpi a)$, and therefore, $\lim \sup_{n\longrightarrow\infty}\sigma(a_n, \varpi u) = 0$. Thus, we have

$$\Omega(\sigma(u, \varpi u)) \leq \Omega\left(\limsup_{n\longrightarrow\infty}\sigma(a_n, \varpi u)\right) + p = -\infty, \qquad (29)$$

which gives $\sigma(u, \varpi u) = 0$, that is, $\varpi u = u$, and u is a fixed point of ϖ.

If v is a fixed point of ϖ in Δ with $\sigma(u, v) < \infty$, then we have $\sigma(u, v) = \sigma(\varpi u, \varpi v) \leq \chi[\sigma(u, \varpi v) + \sigma(v, \varpi u)] = 2\chi\sigma(u, v)$. Consequently, $\sigma(u, v) = 0$, that is, $u = v$.

4. An Application to the System of Linear Algebraic Equations

An application of Theorem 23 for solving a system of linear algebraic equations has been presented in this section.

Consider the following system of n linear algebraic equations with n unknowns:

$$\begin{aligned} p_{11}\lambda_1 + p_{12}\lambda_2 + \cdots + p_{1n}\lambda_n + c_1 &= 0, \\ p_{21}\lambda_1 + p_{22}\lambda_2 + \cdots + p_{2n}\lambda_n + c_2 &= 0, \\ &\vdots \\ p_{n1}\lambda_1 + p_{n2}\lambda_2 + \cdots + p_{nn}\lambda_n + c_n &= 0, \end{aligned} \qquad (30)$$

where $p_{ij}, c_i \in \mathbb{R}$ for all $1 \leq i, j \leq n$. We can write the system of linear equations in matrix notation as $P\Delta + C = O$, where $P = (p_{ij})_{n\times n}$, $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, $C = (c_1, c_2, \cdots, c_n)$, and $O = (0, 0, \cdots, 0)$. To find a solution of the system of linear Equations (30), we have to find a fixed point of the mapping $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $g(\Lambda) = Q\Lambda + C$, where $Q = P + I_n$, that is, $Q = (q_{ij})_{n\times n}$ with $q_{ij} = p_{ij}$ if $i \neq j$ and $q_{ii} = p_{ii} + 1$ for all $i = 1, \cdots, n$.

Now, we define $\sigma : (\mathbb{R}^n)^2 \longrightarrow [0,\infty)$ by

$$\sigma(x, y) = \max_{1\leq i\leq n}\left[|\lambda_i - \lambda_i'| + \left(\lambda_i - \lambda_i'\right)^2\right], \text{where } x = (\lambda_i) \text{ and } y = \left(\lambda_i'\right). \qquad (31)$$

Then, σ is a sequential $\mathcal{F}$-metric for $\Omega(t) = \log t$ and $p = \log 2$.

Theorem 29. *If*

$$\sum_{j=1}^{n}|q_{ij}| + \left(\sum_{j=1}^{n} | q_{ij} |\right)^2 \leq \mu < 1 \text{ for all } 1 \leq i \leq n, \qquad (32)$$

then the system of linear Equations (30) has a unique solution in $(\mathbb{R}^n, \sigma)$.

Proof. To find a unique solution of (30), we show that the mapping $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $g(x) = Qx + C$ for all $x \in \mathbb{R}^n$, where $Q = P + I_n$, that is, $Q = (q_{ij})_{n\times n}$ with $q_{ij} = p_{ij}$ if $i \neq j$ and $q_{ii} = p_{ii} + 1$ for all $i = 1, \cdots, n$, satisfies the contractive condition of Theorem 23. Now, for any $x = (\lambda_i)$ and $y = (\lambda_i')$ in $\mathbb{R}^n$, we have

$$\begin{aligned} \sigma(g(x), g(y)) &= \max_{1\leq i\leq n}\left[\left|\sum_{j=1}^{n} q_{ij}\left(\lambda_j - \lambda_j'\right)\right| + \left(\sum_{j=1}^{n} q_{ij}\left(\lambda_j - \lambda_j'\right)\right)^2\right] \\ &\leq \max_{1\leq i\leq n}\left[\sum_{j=1}^{n}\left|q_{ij}\right|\left|\lambda_j - \lambda_j'\right| + \left(\sum_{j=1}^{n}\left|q_{ij}\right|\left|\lambda_j - \lambda_j'\right|\right)^2\right] \\ &\leq \max_{1\leq i\leq n}\left[\sum_{j=1}^{n}\left|q_{ij}\right|\sigma(x, y) + \left(\sum_{j=1}^{n}\left|q_{ij}\right|\sqrt{\sigma(x, y)}\right)^2\right] \\ &\leq \left(\max_{1\leq i\leq n}\sum_{j=1}^{n}\left|q_{ij}\right|\right)\sigma(x, y) + \left(\max_{1\leq i\leq n}\sum_{j=1}^{n}\left|q_{ij}\right|\right)^2\sigma(x, y) \\ &= \left[\left(\max_{1\leq i\leq n}\sum_{j=1}^{n}\left|q_{ij}\right|\right) + \left(\max_{1\leq i\leq n}\sum_{j=1}^{n}\left|q_{ij}\right|\right)^2\right]\sigma(x, y) \\ &\leq \mu\,\sigma(x, y). \end{aligned} \qquad (33)$$

Since $(\mathbb{R}^n, \sigma)$ is complete, therefore, due to Theorem 23, g has a unique fixed point, that is, the system of linear Equations (30) has a unique solution in $\mathbb{R}^n$.

We now give a numerical example in respect of Theorem 23.

Example 30. Let us consider the following system of linear algebraic equations in three variables:

$$\begin{aligned} 0.9\lambda_1 + 0.1\lambda_2 + 0.15\lambda_3 + 1 &= 0, \\ 0.1\lambda_1 + 0.85\lambda_2 + 0.1\lambda_3 + 2 &= 0, \\ 0.1\lambda_1 + 0.05\lambda_2 + 0.8\lambda_3 + 3 &= 0. \end{aligned} \qquad (34)$$

Then, the system of linear algebraic Equations (34) has a unique solution.

Solution. Let $\Lambda = \mathbb{R}^3$ be the SFMS endowed with the metric $\sigma : \Lambda^2 \longrightarrow [0,\infty)$ defined by

$$\sigma(x, y) = \max_{1\leq i\leq 3}\left[\left|\lambda_i - \lambda_i'\right| + \left(\lambda_i - \lambda_i'\right)^2\right], \text{ for all } x = (\lambda_i) \text{ and } y = \left(\lambda_i'\right) \text{ in } \Lambda. \tag{35}$$

We can write the above system of linear algebraic Equations (34) as

$$\begin{aligned} -0.9\lambda_1 - 0.1\lambda_2 - 0.15\lambda_3 - 1 &= 0, \\ -0.1\lambda_1 - 0.85\lambda_2 - 0.1\lambda_3 - 2 &= 0, \\ -0.1\lambda_1 - 0.05\lambda_2 - 0.8\lambda_3 - 3 &= 0. \end{aligned} \tag{36}$$

Here, $p_{11} = -0.9, p_{12} = -0.1, p_{13} = -0.15, p_{21} = -0.1, p_{22} = -0.85, p_{23} = -0.1, p_{31} = -0.1, p_{32} = -0.05, p_{33} = -0.8, c_1 = -1, c_2 = -2$, and $c_3 = -3$.

Thus, $q_{11} = 0.1$, $q_{12} = -0.1$, $q_{13} = -0.15$, $q_{21} = -0.1$, $q_{22} = 0.15$, $q_{23} = -0.1$, $q_{31} = -0.1$, $q_{32} = -0.05$, and $q_{33} = 0.2$. Also, we see that

$$\begin{aligned} \sum_{j=1}^{3}\left|q_{ij}\right| = 0.35, \text{ i.e., } \sum_{j=1}^{3}\left|q_{ij}\right| + \left(\sum_{j=1}^{3}\left|q_{ij}\right|\right)^2 \\ = 0.4725 < 1 \text{ for all } 1 \leq i \leq 3. \end{aligned} \tag{37}$$

Hence, from the Theorem 23, it follows that the system of linear algebraic Equations (34) has a unique solution in $\mathbb{R}^3$, which is given by $\lambda_1 \simeq -0.3018$, $\lambda_2 \simeq -1.894$, and $\lambda_3 \simeq -3.593$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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