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A General Algorithm of the Boundary Integral Method for Solving Laplace's Mixed Boundary Value Problem

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Authors' contributions

This work was carried out in collaboration among all authors. Author RKP designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors PK and DN managed the analyses of the study. Author DN managed the literature searches. All authors read and approved the final manuscript.

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Abstract

Boundary elements have emerged as a powerful alternative to finite elements particularly in cases where better accuracy is required. The most important features of boundary elements however is that it only requires descretization of the surface rather than the volume. Here, A general algorithm of the boundary integral method has been formulated for solving elliptic partial differential equations. The broad applicability of the approach is illustrated with a problem of practical interest giving the solution of the Laplace equation for potential flow with mixed boundary problems. The results and patterns are shown in tables and figures and compared well with Brebbia [1] are found in good agreement.

Keywords: Boundary elements; boundary integral method; laplace equation.

1 Introduction

Finite element technique has been proved inadequate or inefficient in many engineering and physical sciences applications and in some cases very cumbersome to use. Finite element analysis is still a

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comparatively slow process due to the need to define or redefine meshes in the piece or domain under study. This was the motive behind development of boundary element technique which has proved as an alternative to finite element [2,3].

Boundary elements have emerged as a powerful alternative to finite elements particularly in cases where better accuracy is required due to problems such as stress concentration or where the domain extends to infinity. The most important features of boundary elements however is that it only requires descretization of the surface rather than the volume. Hence boundary element methods are easier to use.

Basic boundary integral equations required by this method can be deduced on the consideration of weighted residuals. It can also be deduced with Green's third identity principle. During 1980's, boundary element methods have attracted attention of scientists and engineers. Brebbia [4] has compared this technique with finite difference and finite element techniques in detail and described its usefulness vis-à-vis with others.

2 Review of Litrature

The credit for the development of Boundary Elements Methods goes to Brebbia and his collaborators. It was the work of Brebbia and Dominguez (see [1]) and [5]) where the name 'Boundary Elements Methods' was used for the first time. In these papers, they compared Boundary Elements Methods verses Finite Elements Methods. They also provided techniques as how to apply Boundary Elements Methods to potential problems. Potential flow problems have wide applicability in Mathematics, Engineering Sciences and Aerodynamics. Generally , they are governed by Laplace's equation which one of the important elliptic partial differential equations. Although previously also, some papers appeared regarding application of 'Integral equation methods' specially in solid mechanics like 'Torsion problem'. But they could not draw much attention due to their limited applicability. It was the work of Brebbia and Ferrante [6] which brought this technique into the forefront. A detail study is also given by Brebbia and Wandland [7].

Herein, the technique is applied on a Laplace's boundary value problem with mixed boundary conditions i.e. having Dirichlet and Neumann both types of boundary conditions. However, the technique can be generalized to tackle Poisson and other type of elliptic partial differential equations.

3 Algorithms for BEM

An algorithm has been developed for computing the results by minimizing the residuals and introducing all important factors as per the following standard procedure using fundamental equations. An overview has been given by Xianyun Qin et. al. [8] for singular integrals on 3D boundary elements.

3.1 Basic integral equation

Laplace equation in a domain Ω (two or three dimensional domain) is

$$\nabla^2 u = o \qquad \text{in } \Omega \tag{1}$$

with following conditions on the Γ boundary

(i) 'essential' Conditions of the type $u = \bar{u}$ on Γ_1
(ii) 'Natural' Conditions such as $q = \partial u/\partial n = \bar{q}$ on Γ_2 (2)

Where n is the normal to the boundary $\Gamma = \Gamma_1 + \Gamma_2$ and the bar indicates that these values are known.

In principle, the error introduced in the above equation if the exact (but unknown) values of u and q are replaced by an approximate solution can be minimized by orthogonalizing them with respect to a weighted function u* , with derivatives on the boundary q* = ∂u*/∂n.

In other words if R are the residuals, one can write in general that

$$\left.\begin{aligned} R &= \nabla^2 u \neq o \\ R_1 &= u - \bar{u} \neq 0 \\ R_2 &= q - \bar{q} \neq o \end{aligned}\right\} \qquad (3)$$

Where u and q are approximate values. (the fact that one or more of the residuals may be identically zero does not detract from the generality of the argument.)

The weighting can now be carried out as shown below

$$\int_{\Omega} Ru^{*}\, d\Omega . = \int_{\Gamma_2} R_2 u^{*}\, d\Gamma - \int_{\Gamma_1} R_1 q^{*}\, d\Gamma \qquad (4)$$

Or,

$$\int_{\Omega} \left(\nabla^2 u\right) u^{*}\, d\Omega = \int_{\Gamma_2} \left(q - \bar{q}\right) u^{*}\, d\Gamma - \int_{\Gamma_1} \left(u - \bar{u}\right) q^{*}\, d\Gamma \qquad (5)$$

Integrating by parts the left hand side of this equation gives.

$$-\int_{\Omega} \left\{ \frac{\partial u}{\partial x_k} \frac{\partial u^{*}}{\partial x_k} \right\} d\Omega = -\int_{\Gamma_2} \bar{q} u^{*}\, d\Gamma - \int_{\Gamma_1} q u^{*} .d\Gamma - \int_{\Gamma_1} u q^{*}\, d\Gamma + \int_{\Gamma_1} \bar{u} q^{*}\, d\Gamma \qquad (6)$$

Where k= 1,2,3 and the so called Einstein's summation for repeated indexes has been used. Integrating by parts again the term on the left hand side one obtains,

$$\int_{\Omega} \left(\nabla^2 u^{*}\right) u\, d\Omega = -\int_{\Gamma_2} \bar{q} u^{*}\, d\Gamma - \int_{\Gamma_1} q u^{*}\, d\Gamma + \int_{\Gamma_2} u q^{*}\, d\Gamma + \int_{\Gamma_1} \bar{u} q^{*}\, d\Gamma \qquad (7)$$

The singular function boundary Integral method for elliptic problems have been elaborated by Xenophontos [9] and Christodoulou et. al. [10].

3.2 Fundamental solution

The fundamental solution u* satisfies Laplace's equation and represents the field generated by a concentrated unit charge acting at a point 'i'. The effect of this charge is propagated from i to infinity without any consideration of boundary conditions. Because of this the solution can be written

$$\nabla^2 u^{*} + \Delta^i = 0 \qquad (8)$$

where Δ^i represents a Dirac Delta function which tends to infinity at the point $x=x^i$ and is equal to zero anywhere else. The integral of Δ^i however is equal to one.

The integral of Dirac delta function multiplied by any other function is equal to the value of the latter at the point x^i. Hence

$$\int_{\Omega} u\left(\nabla^{2} u^{*}\right) d\Omega = \int_{\Omega} u\left(-\Delta^{i}\right) d\Omega = -u^{i} \quad \ldots\ldots\ldots \quad (9)$$

Equation (7) can now be written as,

$$u^{i} + \int_{\Gamma_2} uq^{*}\, d\Gamma + \int_{\Gamma_1} \bar{u} q^{*}\, d\Gamma = \int_{\Gamma_2} \bar{q} u^{*}\, d\Gamma + \int_{\Gamma_1} qu^{*}\, d\Gamma \quad \ldots\ldots\ldots \quad (10)$$

For an isotropic three dimensional medium the fundamental solution of equation (8) is

$$u^{*}=1/4\pi r \quad (11)$$

and for a two dimensional isotropic domain ,it is

$$u^{*} = 1/2\pi \ln (1/r) \quad (12)$$

Where r is the distance from the point x^i of application of the delta function to any point under consideration.

3.3 Boundary integral equation

We have now deduced an equation (10) which is valid for any point within the Ω domain. In boundary elements it is usually preferable for computational reasons to apply equation (10) on the boundary and hence we need to find out what happens when the point x^i is on Γ. A simple way to do this is to consider that the point i is on the boundary but the domain itself is augmented by a hemisphere of radius ε (in 3D) as shown in Fig. 1 (for two D the same applies but we will consider a semicircle instead). The point x^i is considered to be at the centre and then the radius ε is taken to zero. The point will then become a boundary point and the resulting expression the specialization of (10) for a point on Γ.

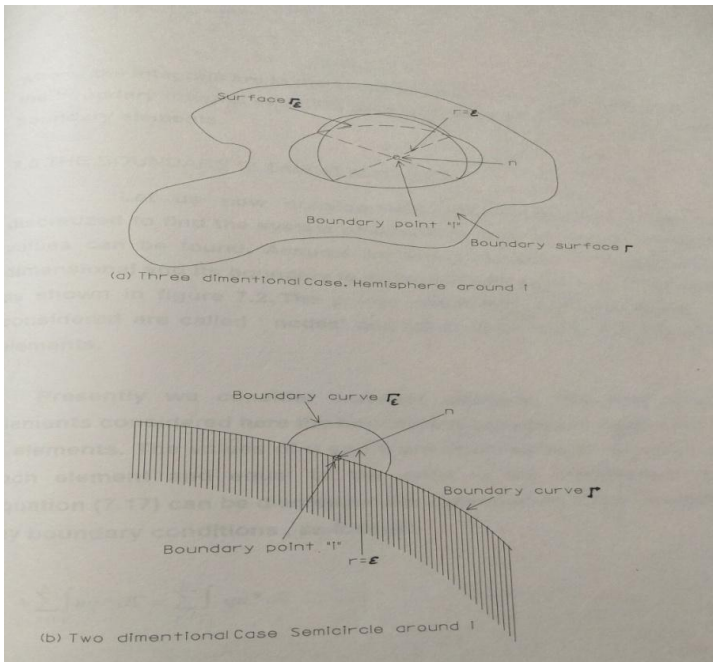

Fig. 1. Boundary points for two and three dimensional case

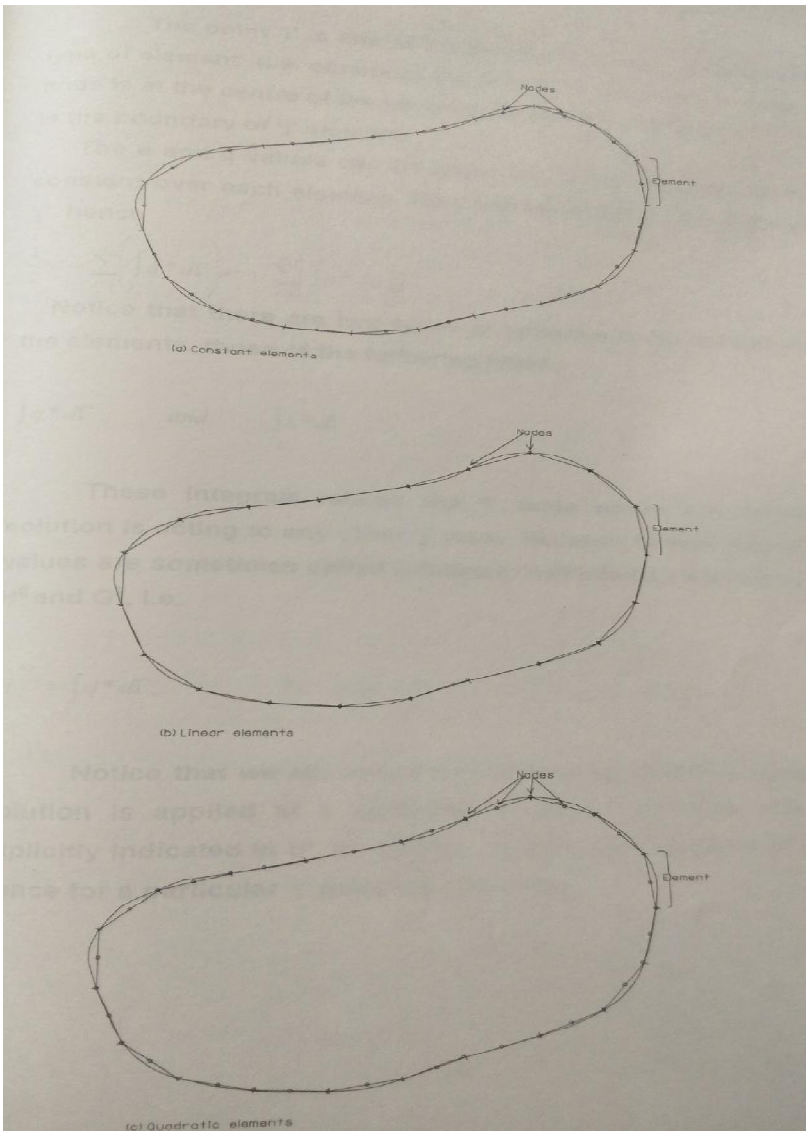

Fig. 2. Different types of boundary elements

It is important at this stage to differentiate between two types of boundary integrals in (10) as the fundamental solution and its derivative behave differently. Consider for the sake of simplicity equation (10) before any boundary conditions have been applied, i.e.

$$u^i + \int_\Gamma uq^* d\Gamma = \int_\Gamma u^* q d\Gamma \quad \text{.....} \quad (13)$$

Here $\Gamma = \Gamma_1 + \Gamma_2$ and satisfaction of the boundary conditions will be left for the latter on. Integrals of the type shown on the right hand side of (13) are easy to deal with as they present a lower order singularity, i.e. for three dimensional cases the integral around Γ_ε gives:

$$\lim_{\varepsilon \to 0} .\left\{ \int_{\Gamma} qu \ * d\Gamma \right\} = \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma} q \frac{1}{4\pi\varepsilon} d\Gamma \right\} = \lim_{\varepsilon \to 0} .\left\{ q \frac{2\pi\varepsilon^2}{4\pi\varepsilon} \right\} = .0 \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

In other words nothing occurs to the right hand side integral when (10) or (13) are taken to the boundary . The left hand side integral however behaves in a different manner . Here we have around Γ_ε the following result,

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma} uq \ * d\Gamma \right\} = \lim_{\varepsilon \to 0} \left\{ - \int_{\Gamma} u \frac{1}{4\pi\varepsilon^2} d\Gamma \right\} = \lim_{\varepsilon \to o} \left\{ - u \frac{2\pi\varepsilon^2}{4\pi\varepsilon^2} \right\} = -\frac{1}{2} u^i \ldots\ldots\ldots\ldots\ldots (15)$$

They produce what is called a free term. It is easy to check that the same will occur for two dimensional problems in which case the right hand side integral around Γ_ε is also identically equal to zero and the left hand side integral becomes,

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma} uq \ * d\Gamma \right\} = \lim_{\varepsilon \to 0} \left\{ - \int_{\Gamma} u \frac{1}{2\pi\varepsilon} d\Gamma \right\} = \lim_{\varepsilon \to 0} \left\{ - u \frac{\pi\varepsilon}{2\pi\varepsilon} \right\} = -\frac{1}{2} u^i \ldots\ldots\ldots\ldots\ldots (16)$$

From (14) to (16) one can write the following expression for two or three dimensional problems

$$\frac{1}{2} u^i + \int_{\Gamma} uq \ * \ d\Gamma = \int_{\Gamma} qu \ * \ d\Gamma \ \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

where the integrals are in the sense of cauchy principal value. This is the boundary integral equation generally used as a starting point for boundary elements.

3.4 The boundary element methods

Let us now consider how expression (17) can be discretized to find the system of equations from which the boundary values can be found. Assume for simplicity that the body is two dimensional and its boundary is divided into N segments or elements as shown in Fig. 2. The points where the unknown values are considered are called 'nodes' and taken to be in the middle of the elements. The numerical solution of diffusion problem in two dimension with variable coefficients is well described by AL-Jawary [11].

3.4.1 BEM for constant elements

For the constant elements considered here the boundary is assumed to be divided into N elements. The values of u and q are assumed to be constant over each element and equal to the value at the mid-element node. Equation (17) can be discretized for a given point 'i' before applying any boundary conditions, as follows,

$$\frac{1}{2} u^i + \sum_{j=1}^{N} \int_{\Gamma} uq \ * d\Gamma = \sum_{j=1}^{N} \int_{\Gamma_j} qu \ * d\Gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

The point 'i' is one of the boundary nodes. Note that for this type of element (i.e. constant) the boundary is always smooth as the node is at the centre of the element, hence the multiplier of u^i is ½. Γ_j is the boundary of 'j' element.

The u and q values can be taken out of the integrals as they are constant over each element. They will be called u^j and q^j for element 'j'. hence

$$\frac{1}{2}u^i + \sum_{j=1}^{N}\left(\int_{\Gamma_j} q^* d\Gamma\right)u^j = \sum_{j=1}^{N}\left(\int_{\Gamma_j} u^* d\Gamma\right)q^j \quad \text{..... ....(19)}$$

Notice that there are two types of integrals to be carried out over the elements, those of the following types,

$$\int_{\Gamma_j} q^* d\Gamma \qquad and \qquad \int_{\Gamma_j} u^* d\Gamma$$

These integrals relates the 'i' node where the fundamental solution is acting to any other 'j' node .Because of this their resulting values are sometimes called influence coefficients. We will call them H^{ij} and G^{ij}, i.e.

$$\overset{\frown}{H}^{i.j} = \int_{\Gamma_j} q^* d\Gamma; \qquad G^{i.j} = \int_{\Gamma_j} u^* d\Gamma \quad \text{..... .....(20)}$$

Notice that we are assuming throughout that the fundamental solution is applied at a particular 'i' node , although this is not explicitly indicated in u*, q* notation to avoid proliferation of indexes. Hence for a particular 'i' point we can write,

$$\frac{1}{2}u^i + \sum_{j=1}^{N}\overset{\frown}{H}^{i.j}u^j = \sum_{j=1}^{N} G^{i.j}q^j \quad \text{..... ....(21)}$$

If we now assume that the position of i can also vary from 1 to N, i.e. we assume that the fundamental solution is applied at each node successively one obtains a system of equations resulting from applying (21) to each boundary point in turn.

Let us now call

$$H^{ij} = \begin{cases} H^{ij} & \text{when } i \neq j \\ H^{ij}+1/2 & \text{when } i=j \end{cases} \qquad (22)$$

Hence equation (21)can now be written as

$$\sum_{j=1}^{N} H^{ij}u^j = \sum_{j=1}^{N} G^{ij}q^j \quad \text{..... ....(23)}$$

This set of matrix equation can be expressed in matrix form as

$$HU=GQ \qquad (24)$$

Where H and G are NxN matrices and U, Q are vectors of length N.

Notice that N_1 values of u and N_2 values of q are known on Γ_1 and Γ_2 respectively ($\Gamma_1+\Gamma_2=\Gamma$), hence there are only N unknowns in system of equations (24). To introduce these boundary conditions into (24) one has

to rearrange the system by moving columns of H and G from one side to the other. Once all the unknowns are passed to the left hand side one can write,

$$AX = F \tag{25}$$

Where X is a vector of unknowns u's and q's boundary values. F is found by multiplying the corresponding columns by the known values of u's or q's. It is interesting to point out that the unknowns are now a mixture of the potential and its derivative, rather than the potential only as in finite elements. This is a consequence of the boundary element being a 'mixed' formulation and gives an important advantage to the method over finite elements.

Equation (25) can now be solved and all the boundary values are then known. Once this is done it is possible to calculate any internal value of u or its derivatives. The values of u's are calculated at any internal point 'i' using formula (10) which can be written as,

$$u^i = \int_\Gamma qu^* \, d\Gamma - \int_\Gamma uq^* \, d\Gamma \quad \text{..... ....} (26)$$

Notice that now the fundamental solution is considered to be acting on an internal point 'i' and that all values of 'u' and 'q' are already known. The process is then one of integration (usually numerically). The same descretization is used for the boundary integrals, i.e.

$$u^i = \sum_{j=1}^{N} G^{ij} q^j - \sum_{j=1}^{N} H^{ij} u^j \quad \text{..... ....} (27)$$

The coefficients G^{ij} and H^{ij} have been calculated a new for each different internal point. The values of internal fluxes in the two directions, say x_1 and x_2, $q_{x1} = \partial u/\partial x_1$ and $q_{x2} = \partial u/\partial x_2$ are calculated by carrying out derivatives on (26), i.e.

$$(q_{x1})^i = \left(\frac{\partial u}{\partial x_1}\right)^i = \int_\Gamma q \left(\frac{\partial u^*}{\partial x_1}\right)^i d\Gamma - \int_\Gamma u \left(\frac{\partial q^*}{\partial x_1}\right)^i d\Gamma \quad \text{..... ....} (28)$$

$$(q_{x2})_i = \left(\frac{\partial u}{\partial x_2}\right)_i = \int_\Gamma q \left(\frac{\partial u^*}{\partial x_2}\right)_i d\Gamma - \int_\Gamma u \left(\frac{\partial q^*}{\partial x_2}\right)_i d\Gamma \quad \text{----------} (29)$$

Notice that the derivatives are carried out only on the fundamental solution functions u* and q* as we are computing the variations of the flux around the 'i' point.

Computation of integrals for internal points in (27), (28) and (29) are usually carried out numerically.

3.4.2 BEM for linear elements

Here we consider linear elements instead of constant elements. We consider a linear variation of u and q for which case the nodes are considered to to be at the ends of the elements as shown in Fig. 3.

The integral equation (13) for linear elements is written as

$$c^i u^i + \int_\Gamma uq^* \, d\Gamma = \int_\Gamma u^* q \, d\Gamma \quad \text{..... ....} (30)$$

After discretizing the boundary into a series of N elements equation (7.30) can be written

$$c^i u^i + \sum_{j=1}^{N} \int_{\Gamma_j} uq^* d\Gamma = \sum_{j=1}^{N} \int_{\Gamma_j} u^* q d\Gamma \quad \text{.....(31)}$$

The integrals in this equation are more difficult to evaluate than those for the constant element as u's and q's are vary linearly over each Γ_j and hence it is not possible to take them out of the integrals.

The values of u and q at any point on the element can be defined in terms of their nodal values and two linear interpolation functions ϕ_1 and ϕ_2, which are given in terms of the homogeneous coordinate ξ as shown in Fig. 3 , i.e.

$$u(\xi) = \phi_1 u^1 + \phi_2 u^2 = [\phi_1 \phi_2]\left[u^1 u^2\right]^T$$
$$q(\xi) = \phi_1 q^1 + \phi_2 q^2 = [\phi_1 \phi_2]\left[q^1 q^2\right]^T \quad \text{.....(32)}$$

ξ is the dimensionless coordinate varying from –1 to +1 and the interpolation function are

$$\phi_1 = \tfrac{1}{2}(1-\xi) \quad : \quad \phi_2 = \tfrac{1}{2}(1+\xi). \qquad (33)$$

let us consider the integrals over an element 'j'. Those on left hand side can be written as,

$$\int_{\Gamma_j} uq^* d\Gamma = \int_{\Gamma_j} [\phi_1 \phi_2] q^* d\Gamma \left[u^1 u^2\right]^T = [h_1^{ij} h_2^{ij}]\left[u^1 u^2\right]^T \quad \text{.....(34)}$$

where for each element 'j' we have the two terms,

$$h_1^{ij} = \int_{\Gamma} \phi_1 q^* d\Gamma \quad \text{.....(35)}$$

and

$$h_2^{ij} = \int_{\Gamma} \phi_2 q^* d\Gamma \quad \text{.....(36)}$$

Similarly the integrals on the right hand side give

$$\int_{\Gamma} qu^* d\Gamma = \int_{\Gamma} [\phi_1 \phi_2] u^* d\Gamma \left[q^1 q^2\right]^T = [g_1^{ij} g_2^{ij}]\left[q^1 q^2\right]^T \quad \text{.....(37)}$$

where

$$g_1^{ij} = \int_{\Gamma} \phi_1 u^* d\Gamma \quad \text{.....(38)}$$

and

$$g_2^{ij} = \int_{\Gamma} \phi_2 u^* d\Gamma \quad \text{.....(39)}$$

Substituting (34) and (37) into (31), the system may be expressed as

$$\sum_{j=1}^{N} H^{i,j} u^j = \sum_{j=1}^{2N} G^{i,j} q^j \quad \text{.....(40)}$$

And the whole set in matrix form becomes

$$H\,U = G\,Q \tag{41}$$

Solving exactly as in the case of constant elements, we get values of unknowns.

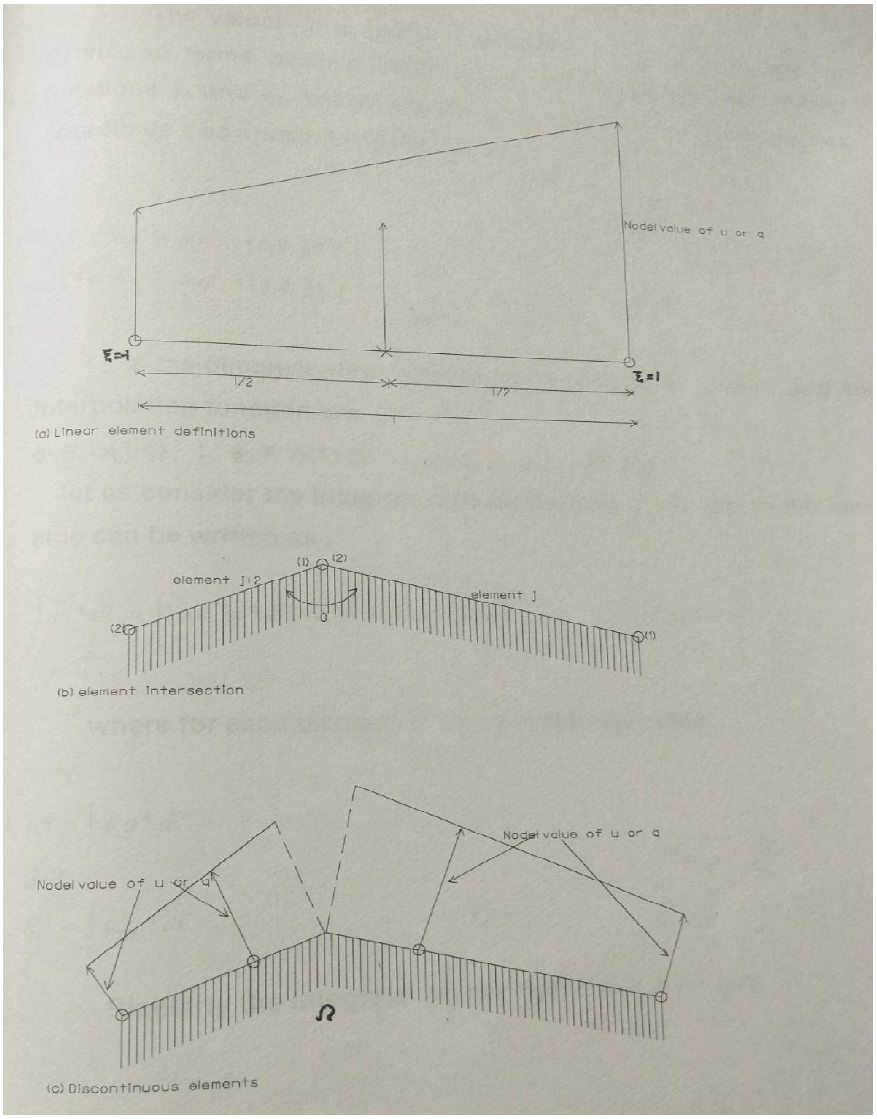

Fig. 3. Linear elements basic definitions and corner treatment

4 The Problem:

Consider the Laplace's equation

$$\nabla^2 u = 0 \tag{42}$$

in a domain Ω which is a square with each side unity as shown in Fig. 4

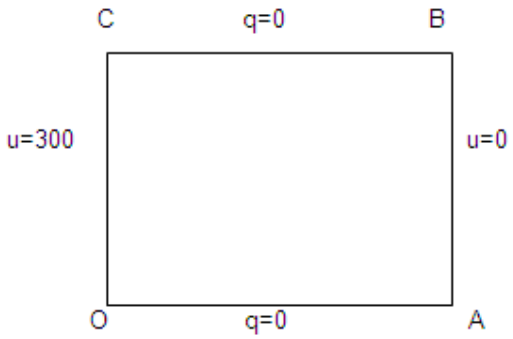

Fig. 4. Domain Ω

Boundary conditions are mixed type as follows

u = 300 along OC
u = 0 along AB
q = 0 along OA and BC

(43)

5 Results and Discussion

The numerical results are obtained using the given algorithm for constant elements by modern computing device and techniques. We discretize the domain into boundary elements and internal nodes as follows:

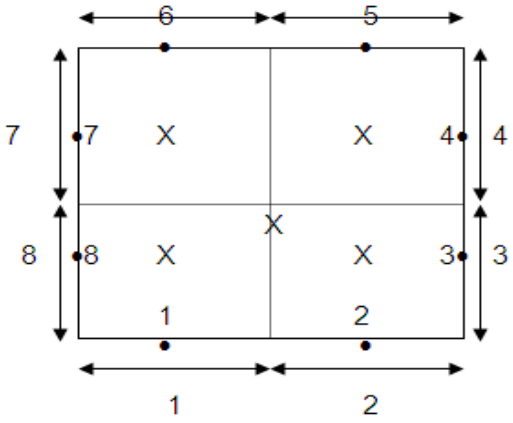

Fig. 5. X →Internal points

Here the domain is descretized into 8 boundary elements and 5 internal nodes. Computational results are depicted in Table 1 for boundary nodes and in Table 2 internal nodes. Pattern of results in Table 1 is in good agreement with given boundary conditions. Moreover, values of potential at internal nodes from Table 2 are shown in Fig. 6

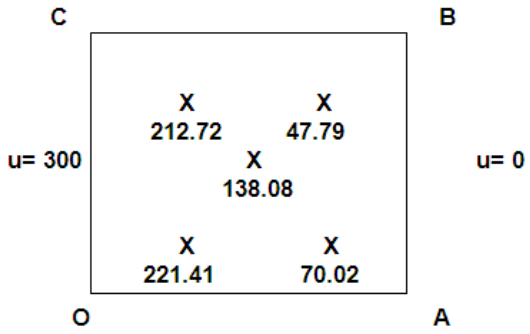

Fig. 6. Values of potential at 5 internal nodes

Above computational results also validate the given boundary conditions, since value of u is zero along AB and 300 along OC.

The same problem is recomputed by taking 16 boundary elements and nine internal nodes as follows:

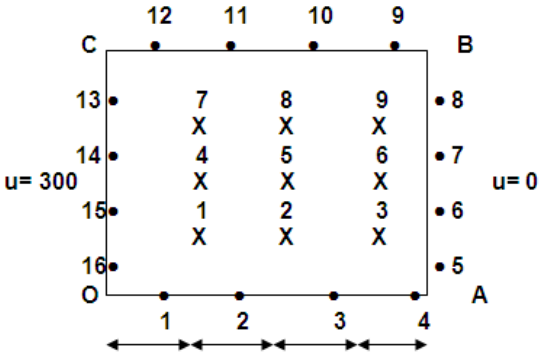

Fig. 7. X Internal nodes

Computational results are given in Table 3 for boundary nodes and in Table 4 for internal nodes. Values of potential u at internal nodes are shown in Fig. 8 here.

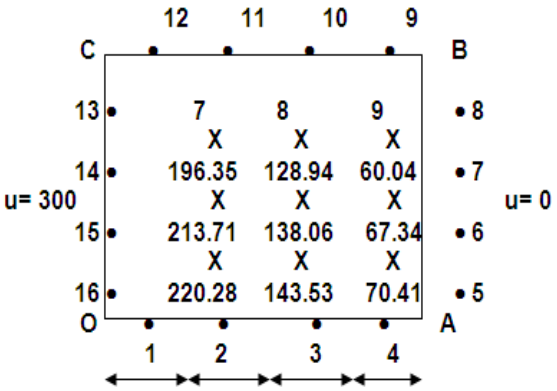

Fig. 8. Values of potential at 9 internal nodes

Results are in excellent agreement with given boundary conditions as well as with results obtained by 8 elements and 5 internal nodes.

From Tables 1-4, it is apparent that results using constant elements have same pattern and features. The same conclusion was expected by Brebbia [1]. Pattern of computational results at lower boundary OA is shown in Fig. 9 for constant elements.

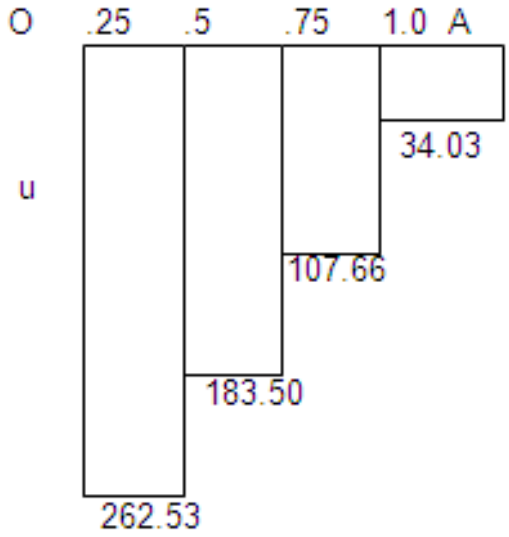

Fig. 9.

Table 1. Value of potential and potential derivatives taking 8 Bounday nodes

Boundary nodes	X	Y	Potential	Potential derivative
1	0.25	0.00	224.62	0.000
2	0.75	0.00	69.284	0.000
3	1.00	0.25	0.000	-303.36
4	1.00	0.75	0.000	-129.40
5	0.75	1.00	0.000	-275.76
6	0.25	1.00	210.67	0.000
7	0.00	0.75	300.00	373.80
8	0.00	0.25	300.00	334.93

Table 2. Values of potential taking 5 internal nodes

Internal nodes	X	Y	Potential
1	0.25	0.25	221.41
2	0.75	0.25	70.02
3	0.50	0.50	138.08
4	0.25	0.75	212.72
5	0.75	0.75	47.79

Table 3. Value of potential and potential derivatives taking 16 boundary nodes

Boundary nodes	X	Y	Potential	Potential derivative
1	0.125	0.000	262.53	0.00
2	0.375	0.000	183.50	0.00
3	0.625	0.000	107.66	0.00
4	0.875	0.000	340.36	0.00
5	1.000	0.125	0.00	-301.48
6	1.000	0.375	0.00	-271.77
7	1.000	0.625	0.00	-254.92
8	1.000	0.875	0.00	-114.41
9	0.875	1.000	0.00	-231.65
10	0.625	1.000	86.68	0.00
11	0.375	1.000	155.93	0.00
12	0.125	1.000	205.42	0.00
13	0.000	0.875	220.48	0.00
14	0.000	0.625	300.00	506.37
15	0.000	0.375	300.00	322.48
16	0.000	0.125	300.00	334.92

Table 4. Values of potential taking 9 internal nodes

Internal nodes	X	Y	Potential
1	0.25	0.25	220.28
2	0.50	0.25	143.53
3	0.75	0.25	70.41
4	0.25	0.50	213.71
5	0.50	0.50	138.06
6	0.75	0.50	67.35
7	0.25	0.75	196.35
8	0.50	0.75	128.94
9	0.75	0.75	60.04

6 Conclusions

Above potential problem governed by Laplace's equation clearly demonstrates the applicability of Boundary Element Methods. It is also clear that Boundary Element Methods are easy to apply and flexible in approach. It is also obvious that by increasing number of elements and nodes give better results according to expectation. The same concept can also be extended for other types of elliptic partial differential equations.

Competing Interests

Authors have declared that no competing interests exist.

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