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A New Common Fixed-Point Theorem for Two Pairs of Mappings in Parametric Metric Space

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Authors' contributions

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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Abstract

Our aim of this paper is to prove a new general common fixed point theorem for two pair of mappings under a different set of conditions using the idea of weakly compatible mappings satisfying a general class of contractions defined by an implicit relation in the frame work of parametric metric space, which unify, extend and generalize most of the existing relevant common fixed point theorems from the literature. Some related results and illustrative an example to highlight the realized improvements is also furnished.

Keywords: Parametric metric space; common fixed point; implicit relation; weakly compatible mappings; contractions.

1 Introduction and Preliminaries

Fixed point theory has attracted many researchers since 1922 with the admired Banach fixed point theorem (see [1]). Banach's contraction principle is one of the pivotal results of analysis. Its significance lies in its

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vast applicability to a great number of branches of mathematics and other sciences, for example, theory of existence of solutions for nonlinear differential, integral, and functional equations, variational inequalities, and optimization and approximation theory. A huge literature on this subject exists and this is a very active area of research at present. Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. The notion of parametric metric spaces being a natural generalization of metric spaces was recently introduced and studied by Hussain et al. [2].

The following definitions are required in the sequel which can be found in [2].

Definition 1.1: Let X be a nonempty set and $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on X if,

1) $\mathcal{P}(x, y, t) = 0, \forall t > 0$ if and only if $x = y$;
2) $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t) \;\; \forall x, y \in X$ and $t > 0$;
3) $\mathcal{P}(x, y, t) \leq \mathcal{P}(x, z, t) + \mathcal{P}(z, y, t) \; \forall x, y, z \in X$ and $t > 0$:

and one says the pair $(X, \mathcal{P})$ is a parametric metric space.

The following definitions are required in the sequel which can be found in [2].

Definition 1.2: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(\mathrm{X}, \mathcal{P})$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \mathcal{P}(x_n, x, t) = 0$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0$.
3. $(\mathrm{X}, \mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 1.3: Let $(\mathrm{X}, \mathcal{P})$ be a parametric metric space and $T: X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.

Example1.4: Let X denote the set of all functions $f : (0, +\infty) \to \mathbb{R}$. Define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by $\mathcal{P}(f, g, t) = |f(t) - g(t)| \; \forall f, g \in X$ and all $t > 0$. Then $\mathcal{P}$ is a parametric metric on X and the pair $(X, \mathcal{P})$ is a parametric metric space.

The following definitions will be needed in the sequel.

Definition 1.5: (see [3]) Let F and G be two self-mappings on a nonempty set X. Then F and G are said to be weakly compatible if they commute at all their coincidence points; that is, $F\omega = G\omega$ for some $\omega \in X$ and then $FG\omega = GF\omega$.

Definition 1.6: (see [4]) Two finite families of self-mappings $\{F_i\}_{i=1}^{m}$ and $\{G_k\}_{k=1}^{n}$ of a non-empty set X are said to be pairwise commuting if

1. $F_iF_j = F_jF_i, \forall \, i, j \in \{1,2, \dots, m\}$
2. $G_kG_l = G_lG_k, \forall \, k, l \in \{1,2, \dots, p\}$
3. $F_iG_k = G_kF_i, \forall \, i \in \{1,2, \dots, m\}$ and $k \in \{1,2, \dots, p\}$.

Lemma 1.7: (see [5]) Let F, G, and f be self-mappings on a nonempty set X with F, G, and f having a unique point of coincidence in X. If (F, f) and (G, f) are weakly compatible. Then F, G and f have a unique common fixed point.

Implicit relations: Simple and natural way to unify and prove in a simple manner several metrical fixed-point theorems are to consider an implicit contraction type condition instead of the usual explicit contractive conditions. Popa [6,7] proved several fixed-point theorems satisfying suitable implicit relations. For proving such results, Popa [6,7] considered Ψ to be the set of all continuous functions

$$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6 \to \mathbb{R}$$

satisfying the following conditions:

(ψ_1). ψ is non-increasing in variables ξ_5 and ξ_6.
(ψ_2). there exists $k \in (0,1)$ such that for $\xi,\zeta \geq 0$ with
(ψ_{2a}). $\psi(\xi,\zeta,\zeta,\xi,\xi+\zeta,0) \leq 0$ or
(ψ_{2b}). $\psi(\xi,\zeta,\xi,\zeta,0,\xi+\zeta) \leq 0 \Rightarrow \xi \leq k\zeta$,
(ψ_3). $\psi(\xi,\xi,0,0,\xi,\xi) > 0$.

Some of the following examples of such functions ψ satisfying (ψ_1), (ψ_2) and (ψ_3) are taken from Popa [7], Imdad and Ali [8] and Berinde [9].

Example 1.8: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6 \to \mathbb{R}$ as:

(1) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1 - k\max\left\{\xi_2,\xi_3,\xi_4,\frac{1}{2}(\xi_5+\xi_6)\right\}$$

where $k \in (0,1)$.

(2) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1^2 - \xi_1(a\xi_2+b\xi_3+c\xi_4) - d\xi_5\xi_6$$

where $a > 0,\ b,c,d \geq 0, a+b+c < 1, a+d < 1$.

(3) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1^3 - a\xi_1^2\xi_2 - b\xi_1\xi_2\xi_3 + c\xi_5^2\xi_6 - d\xi_5\xi_6^2$$

where $a > 0, b,c,d \geq 0, a+b < 1, a+c+d < 1$.

(4) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1^3 - k\left(\frac{\xi_3^2\xi_4^2+\xi_5^2\xi_6^2}{\xi_2+\xi_3+\xi_4+1}\right)$$

where $k \in (0,1)$.

(5) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1^2 - a\xi_2^2 - b\left(\frac{\xi_5\xi_6}{\xi_3^2+\xi_4^2+1}\right)$$

Where $a > 0, b \geq 0, a+b < 1$.

(6) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1^2 - a\max\{\xi_2^2\xi_3^2\xi_4^2\} - b\max\{\xi_3\xi_5,\xi_4\xi_6\} - c\xi_5\xi_6$$

where $a > 0, b,c \geq 0, a+2b < 1, a+c < 1$.

(7) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1 - k\max\left\{\xi_2,\xi_3,\xi_4,\frac{1}{2}\xi_5,\frac{1}{2}\xi_6\right\}$$

where $k \in (0,1)$.

(8) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1 - k\max\left\{\xi_2,\frac{\xi_3+\xi_4}{2},\frac{\xi_5+\xi_6}{2}\right\}$$

Where $k \in (0,1)$.

(9) $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\xi_1-(a\xi_2+b\xi_3+c\xi_4+d\xi_5+c\xi_6)$

where $d,e\geq 0, a+b+c+d+e<1$.

(10) $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\xi_1-\frac{k}{2}\max\{\xi_2,\xi_3,\xi_4,\xi_5,\xi_6\}$

where $k\in(0,1)$.

(11) $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\xi_1-[a\xi_2+b\xi_3+c\xi_4+d(\xi_5+\xi_6)]$

where $d\geq 0, a+b+c+2d<1$.

Since verifications of requirements $(\psi_1),(\psi_2)$ and (ψ_3) for Examples (1)-(11) are straightforward, hence details are omitted. Here one may further notice that some other well-known contraction conditions [10,11,12] can also be deduced as particular cases of implicit relation of Popa [7]. In order to strengthen this viewpoint, we add some more examples to this effect and utilize them to demonstrate how this implicit relation can cover several other known contractive conditions and is also good enough to yield further unknown natural contractive conditions as well.

Example 1.9: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6\to\mathbb{R}$ as:

(12) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\begin{cases}\xi_1-a_1\frac{\xi_3^2+\xi_4^2}{\xi_3+\xi_4}-a_2\xi_2-a_3(\xi_5+\xi_6), if\ \xi_3+\xi_4\neq 0,\\ \xi_1, \quad if\ \xi_3+\xi_4=0.\end{cases}$$

where $a_i\geq 0$ $(i=1,2,3)$ with at least one a_i non-zero and $a_1+a_2+2a_3<1$. (ψ_1). Obviously, ψ is non-increasing in variables ξ_5 and ξ_6. (ψ_{2a}). Let $\xi>0$. Then

$$\psi(\xi,\zeta,\zeta,\xi,\xi+\zeta,0)=\xi-a_1\frac{\zeta^2+\xi^2}{\zeta+\xi}-a_2\zeta-a_3(\xi+\zeta)\leq 0.$$

If $\xi\geq\zeta$, then

$$\xi\leq(a_1+a_2+2a_3)\xi<\xi$$

which is contradiction. Hence $\xi<\zeta$ and $\xi\leq k\zeta$ where $k\in(0,1)$. (ψ_{2b}). Similar argument as in (ψ_{2a}). (ψ_3). $\psi(\xi,\xi,0,0,\xi,\xi)=\xi>0$ for all $\xi>0$.

Example 1.10: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6\to\mathbb{R}$ as:

(13) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\begin{cases}\xi_1-a_1\xi_2-\frac{a_2\xi_3\xi_4+a_3\xi_5\xi_6}{\xi_3+\xi_4},\ if\ \xi_3+\xi_4\neq 0,\\ \xi_1, \quad if\ \xi_3+\xi_4=0.\end{cases}$$

where $a_1,a_2,a_3\geq 0$ such that $1<2a_1+a_2<2$.

Example 1.11: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6\to\mathbb{R}$ as:

(14) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\xi_1-a_1\Big[a_2\max\Big\{\xi_2,\xi_3,\xi_4,\frac{1}{2}(\xi_5+\xi_6)\Big\}$$
$$+(1-a_2)\Big[\max\Big\{\xi_2^1,\xi_3\xi_4,\xi_5\xi_6,\frac{\xi_3\xi_6}{2},\frac{\xi_4\xi_5}{2}\Big\}\Big]^{\frac{1}{2}}\Big]$$

where $a_1\in(0,1)$ and $0\leq a_2\leq 1$.

Example 1.12: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6 \to \mathbb{R}$ as:

(15) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_2^1 - a_1 \max\{\xi_2^2,\xi_3^2,\xi_4^2\} - a_2\, max\left\{\frac{\xi_3\xi_6}{2},\frac{\xi_4\xi_5}{2}\right\} - a_3\,\xi_5\xi_6$$

where $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$.

Very recently, Popa et al. [13] proved several fixed point theorems satisfying suitable implicit relations in which Husain and Sehgal [14] type contraction conditions [15,16,17,18] can be deduced from similar implicit relations in addition to all earlier ones if there is a slight modification in condition (ψ_1) as follows:

$(\Psi_1)'$Obviously, ψ is decreasing in variables $\xi_2, \ldots\ldots, \xi_6$.

Hereafter, let $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) : \mathbb{R}_+^6 \to \mathbb{R}$ be a continuous function which satisfies the conditions $(\psi_1)'$, (ψ_2) and (ψ_3) and $\mathcal{F}$ be the family of such functions. In this paper, we employ such implicit relation to prove our results. But before we proceed further, let us furnish some examples to highlight the utility of the modifications instrumented herein.

Example 1.13: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6 \to \mathbb{R}$ as:

(16) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1 - \phi\left(\max\left\{\xi_2,\xi_3,\xi_4,\tfrac{1}{2}(\xi_5+\xi_6)\right\}\right)$$

where $\phi:\mathbb{R}^+ \to \mathbb{R}^+$ is an increasing upper semi-continuous function with $\phi(0) = 0$ and $\phi(\xi) < \xi$ for each $\xi > 0$.

$(\psi_1)'$ Obviously, ψ is decreasing in variables $\xi_2, \ldots\ldots, \xi_6$.

(ψ_{2a}). Let $\xi > 0$. Then

$$\psi(\xi,\zeta,\zeta,\xi,\xi+\zeta,0) = \xi - \phi\left(\max\left\{\zeta,\zeta,\xi,\frac{1}{2}(\zeta+\xi)\right\}\right) < 0.$$

If $\xi \geq \zeta$, then

$$\xi \leq \phi(\xi) < \xi$$

which is contradiction. Hence $\xi < \zeta$ and $\xi \leq k\zeta$ where $k \in (0,1)$.

(Ψ_{2b}). Similar argument as in (ψ_{2a}).

(Ψ_3). $\psi(\xi,\xi,0,0,\xi,\xi) = \xi - \phi\left(\max\left\{\xi,0,0,\tfrac{1}{2}(\xi+\xi)\right\}\right) = \xi - \phi(\xi) > 0$ for all $\xi > 0$.

Example 1.14: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6 \to \mathbb{R}$ as:

(17) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1 - \phi(\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)$$

where $\phi:\mathbb{R}_+^5 \to \mathbb{R}^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\phi(\xi,\xi,a\xi,b\xi,c\xi) < \xi$ for each $\xi > 0$ and $a,b,c \geq 0$ with $a+b+c \leq 3$.

Example 1.15: Define $\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6):\mathbb{R}_+^6 \to \mathbb{R}$ as:

(18) $$\psi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \xi_1^2 - \phi(\xi_2^2,\xi_3\xi_4,\xi_5\xi_6,\xi_3\xi_6,\xi_4\xi_5),$$

where $\phi: \mathbb{R}_+^5 \to \mathbb{R}^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\phi(\xi, \xi, a\xi, b\xi, c\xi) < \xi$ for each $\xi > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Jungck [3] proved the interesting generalization of celebrated Banach contraction principle. While proving his result, Jungck [3] replaced identity map with a continuous mapping. In [17], Imdad and Ali established a general common fixed-point theorem for a pair of mappings using a suitable implicit function without the requirement of the containment of ranges.

In this paper, we establish a new general common fixed point theorem for two pair of mappings under a different set of conditions using the idea of weakly compatible mappings satisfying a general class of contractions defined by an implicit relation in the frame work of parametric metric space, which unify, extend and generalize most of the existing relevant common fixed point theorems from the literature. Some related results and illustrative an example to highlight the realized improvements is also furnished.

2 Main Results

The following theorem is our main result.

Theorem 2.1: Let F, G, f and g be four self-maps of a parametric metric space $(X, \mathcal{P})$ with $\overline{G(X)} \subseteq f(X)$ and $\overline{F(X)} \subseteq g(X)$ and for all $x, y \in X$, $t > 0$ and some $\psi \in \Psi$,

$$\psi\big(\mathcal{P}(Fx, Gy, t), \mathcal{P}(fx, gy, t), \mathcal{P}(fx, Fx, t), \mathcal{P}(gy, Gy, t), \mathcal{P}(fx, Gy, t), \mathcal{P}(Fx, gy, t)\big) \leq 0 \tag{2.1}$$

If one of $\overline{G(X)}$ and $\overline{F(X)}$ is a complete subspace of X, then (F, f) and (G, g) have a unique point of coincidence inX. Moreover, if (F, f) and (G, g) are weakly compatible, then F, G, f and g have a unique common fixed point in X.

Proof Let $x_0 \in X$ be arbitrary point. Because $G(X) \subseteq \overline{G(X)}$ and $F(X) \subseteq \overline{F(X)}$, we have $F(X) \subseteq f(X)$ and $(X) \subseteq g(X)$. Hence one can inductively define the sequences $\{x_n\} \subset X$ and $\{y_n\} \subset X$ in the following way:

$$y_{2n-1} = Fx_{2n-1} = gx_{2n},$$

$$y_{2n} = Gx_{2n} = fx_{2n+1}, \ \forall\, n \in \mathbb{N}. \tag{2.2}$$

From (2.1) with $x = x_{2n+1}$ and $y = x_{2n+2}$, we get for all $t > 0$ and all $n \in \mathbb{N}$,

$$\psi(\mathcal{P}(Fx_{2n+1}, Gx_{2n+2}, t), \mathcal{P}(fx_{2n+1}, gx_{2n+2}, t), \mathcal{P}(fx_{2n+1}, Fx_{2n+1}, t),$$

$$\mathcal{P}(gx_{2n+2}, Gx_{2n+2}, t), \mathcal{P}(fx_{2n+1}, Gx_{2n+2}, t), \mathcal{P}(Fx_{2n+1}, gx_{2n+2}, t)\big) \leq 0 \tag{2.3}$$

we have

$$\psi(\mathcal{P}(y_{2n+1}, y_{2n+2}, t), \mathcal{P}(y_{2n}, y_{2n+1}, t), \mathcal{P}(y_{2n}, y_{2n+1}, t),$$

$$\mathcal{P}(y_{2n+1}, y_{2n+2}, t), \mathcal{P}(y_{2n}, y_{2n+2}, t), \mathcal{P}(y_{2n+1}, y_{2n+1}, t)\big) \leq 0,$$

That is,

$$\psi(\mathcal{P}(y_{2n+1}, y_{2n+2}, t), \mathcal{P}(y_{2n}, y_{2n+1}, t), \mathcal{P}(y_{2n}, y_{2n+1}, t),$$

$$\mathcal{P}(y_{2n+1}, y_{2n+2}, t), \mathcal{P}(y_{2n}, y_{2n+2}, t), 0) \leq 0, \tag{2.4}$$

Using the fact that ψ is non-increasing in variable u_5 and u_6, we have

$$\mathcal{P}(y_{2n}, y_{2n+2}, t) \leq \mathcal{P}(y_{2n}, y_{2n+1}, t) + \mathcal{P}(y_{2n+1}, y_{2n+2}, t) \tag{2.5}$$

From (2.4), we derive that

$$\psi(\mathcal{P}(y_{2n+1}, y_{2n+2}, t), \mathcal{P}(y_{2n}, y_{2n+1}, t), \mathcal{P}(y_{2n}, \quad y_{2n+1}, t),$$
$$\mathcal{P}(y_{2n+1}, y_{2n+2}, t), \mathcal{P}(y_{2n}, y_{2n+1}, t) + \mathcal{P}(y_{2n+1}, y_{2n+2}, t), 0) \leq 0, \tag{2.6}$$

Now, using property (ψ_{2a}), we have

$$\mathcal{P}(y_{2n+1}, y_{2n+2}, t) \leq k\mathcal{P}(y_{2n}, y_{2n+1}, t). \tag{2.7}$$

Again, using (2.1), with $x = x_{2n}$ and $y = x_{2n+1}$, we get for all $t > 0$ and all $n \in \mathbb{N}$,

$$\psi(\mathcal{P}(Fx_{2n}, Gx_{2n+1}, t), \mathcal{P}(fx_{2n}, gx_{2n+1}, t), \mathcal{P}(fx_{2n}, Fx_{2n}, t),$$
$$\mathcal{P}(gx_{2n+1}, Gx_{2n+1}, t), \mathcal{P}(fx_{2n}, Gx_{2n+1}, t), \mathcal{P}(Fx_{2n}, gx_{2n+1}, t)) \leq 0, \tag{2.8}$$

That is,

$$\psi(\mathcal{P}(y_{2n}, y_{2n+1}, t), \mathcal{P}(y_{2n-1}, y_{2n}, t), \mathcal{P}(y_{2n-1}, y_{2n}, t),$$
$$\mathcal{P}(y_{2n}, y_{2n+1}, t), \mathcal{P}(y_{2n-1}, y_{2n+1}, t), 0) \leq 0, \tag{2.9}$$

Keeping in mind that ψ is non-increasing in variable u_5 and u_6, we have

$$\mathcal{P}(y_{2n-1}, y_{2n+1}, t) \leq \mathcal{P}(y_{2n-1}, y_{2n}, t) + \mathcal{P}(y_{2n}, y_{2n+1}, t) \tag{2.10}$$

From (2.9), we obtain

$$\psi(\mathcal{P}(y_{2n}, y_{2n+1}, t), \mathcal{P}(y_{2n-1}, y_{2n}, t), \mathcal{P}(y_{2n-1}, y_{2n}, t),$$
$$\mathcal{P}(y_{2n}, y_{2n+1}, t), \mathcal{P}(y_{2n-1}, y_{2n}, t) + \mathcal{P}(y_{2n}, y_{2n+1}, t), 0) \leq 0, \tag{2.11}$$

yielding thereby (due to (ψ_{2a})),

$$\mathcal{P}(y_{2n}, y_{2n+1}, t) \leq k\mathcal{P}(y_{2n-1}, y_{2n}, t). \tag{2.12}$$

Combining (2.7) and (2.12), we have

$$\mathcal{P}(y_{2n+1}, y_{2n+2}, t) \leq k^2\mathcal{P}(y_{2n-1}, y_{2n}, t) \tag{2.13}$$

Now by induction, we obtain for each $n = 0, 1, 2, \ldots$

$$\mathcal{P}(y_{2n+1}, y_{2n+2}, t) \leq k\mathcal{P}(y_{2n}, y_{2n+1}, t)$$
$$\leq \cdots \leq k^{2n+1}\mathcal{P}(y_0, y_1, t). \tag{2.14}$$

and by a routine calculation, we have,

$$\mathcal{P}(y_{n+1}, y_{n+2}, t) \leq k\mathcal{P}(y_n, y_{n+1}, t)$$
$$\leq \cdots \leq k^{n+1}\mathcal{P}(y_0, y_1, t). \tag{2.15}$$

Hence for each $n > m$, we obtain

$$\mathcal{P}(y_n, y_m, t) \leq \mathcal{P}(y_n, y_{n-1}, t) + \mathcal{P}(y_{n-1}, y_{n-2}, t) + \cdots + \mathcal{P}(y_{m+1}, y_m, t)$$

$$\leq (k^{n-1} + k^{n-2} + \cdots + k^m)\mathcal{P}(y_0, y_1, t)$$

$$\leq \frac{k^m}{1-k}\mathcal{P}(y_0, y_1, t) \tag{2.16}$$

Therefore, $\{y_n\}$ is a Cauchy sequence. Assume that $\overline{G(X)}$ is complete. Observe that the subsequence $\{y_{2n}\}$ is a Cauchy sequence which is contained in $\overline{G(X)}$ must a limit ω^* in $f(X)$, that is,

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Gx_{2n} = \lim_{n\to\infty} fx_{2n+1} \in \overline{G(X)} \subseteq f(X) \subset X,$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Gx_{2n} = \lim_{n\to\infty} fx_{2n+1} = \omega^* \in f(X). \tag{2.17}$$

It is easy to see

$$\omega^* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Gx_{2n} = \lim_{n\to\infty} f\mathrm{x}_{2n+1} = \lim_{n\to\infty} Gx_{2n}$$

$$= \lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} Fx_{2n-1} = \lim_{n\to\infty} gx_{2n} \tag{2.18}$$

Consequently, we can find $\omega \in X$ such that $f\omega = \omega^*$. We assert that $F\omega = f\omega = \omega^*$. If not, then $\mathcal{P}(F\omega, \omega^*, t) > 0$. Using (2.1), with $x = \omega$ and $y = x_{2n}$, we have

$$\psi(\mathcal{P}(F\omega, Gx_{2n}, t), \mathcal{P}(f\omega, gx_{2n}, t), \mathcal{P}(f\omega, F\omega, t),$$

$$\mathcal{P}(gx_{2n}, Gx_{2n}, t), \mathcal{P}(f\omega, Gx_{2n}, t), \mathcal{P}(\mathrm{S}\omega, gx_{2n}, t)\big) \leq 0$$

$$\Rightarrow \quad \psi(\mathcal{P}(F\omega, y_{2n}, t), \mathcal{P}(f\omega, \quad y_{2n-1}, t), \mathcal{P}(f\omega, \quad F\omega, t),$$

$$\mathcal{P}(y_{2n-1}, y_{2n}, t), \mathcal{P}(f\omega, y_{2n}, t), \mathcal{P}(F\omega, y_{2n-1}, t)\big) \leq 0, \tag{2.19}$$

Letting $n \to +\infty$ in the above inequality, using (2.18) and the continuity of ψ, we have

$$\psi(\mathcal{P}(F\omega, \omega^*, t), 0, \mathcal{P}(\omega^*, \ F\omega, t), 0, 0, \mathcal{P}(F\omega, \omega^*, t)\big) \leq 0, \tag{2.20}$$

yielding thereby $\big($due to $(\psi_{2b})\big)$, $\mathcal{P}(F\omega, \omega^*, t) \leq 0$, that is $\mathcal{P}(F\omega, \omega^*, t) = 0$, which is a contradiction. Then we have $F\omega = f\omega = \omega^*$, which shows that ω is a coincidence point of F and f, that is ω^* is a point of coincidence of F and f. Since $\omega^* = F\omega \in F(X) \subseteq \overline{F(X)} \subseteq g(X)$, there exists $\omega' \in X$ such that $g\omega' = \omega^*$. We claim that $G\omega' = \omega^*$. If not, then $\mathcal{P}(G\omega', \omega^*, t) > 0$. Using (2.1), with $x = \omega$ and $y = \omega'$, we have

$$\psi(\mathcal{P}(F\omega, G\omega', t), \mathcal{P}(f\omega, g\omega', t), \mathcal{P}(f\omega, F\omega, t),$$

$$\mathcal{P}(g\omega', G\omega', t), \mathcal{P}(f\omega, G\omega', t), \mathcal{P}(F\omega, g\omega', t)\big) \leq 0,$$

That is,

$$\psi(\mathcal{P}(\omega^*, G\omega', t), 0, 0, \mathcal{P}(\omega^*, G\omega', t), \mathcal{P}(\omega^*, G\omega', t), 0) \leq 0, \tag{2.21}$$

yielding thereby (due to (ψ_{2a})), $\mathcal{P}(\omega^*, G\omega', t) \leq 0$, then $\mathcal{P}(\omega^*, G\omega', t) = 0$. Thus, our supposition that $\mathcal{P}(G\omega', \omega^*, t) > 0$ was wrong. Therefore $G\omega' = g\omega' = \omega^*$, which shows that ω' is a coincidence point of G and g, that is ω^* is a point of coincidence of G and g. Now, suppose that ω_* is another point of coincidence of F and f, that is $\omega_* = F\varpi = f\varpi$ for some $\varpi \in X$. Using (2.1), we have

$$\psi(\mathcal{P}(F\varpi, G\omega', t), \mathcal{P}(f\varpi, g\omega', t), \mathcal{P}(f\varpi, F\varpi, t),$$

$$\mathcal{P}(g\omega', G\omega', t), \mathcal{P}(f\varpi, G\omega', t), \mathcal{P}(F\varpi, g\omega', t)) \leq 0, \tag{2.22}$$

This implies that

$$\psi\big(\mathcal{P}(\omega_*, \omega^*, t), \mathcal{P}(\omega_*, \omega^*, t), \mathcal{P}(\omega_*, \omega_*, t), \mathcal{P}(\omega^*, \omega^*, t), \mathcal{P}(\omega_*, \omega^*, t), \mathcal{P}(\omega_*, \omega^*, t)\big) \leq 0$$

That is,

$$\psi\big(\mathcal{P}(\omega_*, \omega^*, t), \mathcal{P}(\omega_*, \omega^*, t), 0, 0, \mathcal{P}(\omega_*, \omega^*, t), \mathcal{P}(\omega_*, \omega^*, t)\big) \leq 0 \tag{2.23}$$

Due to (ψ_3), we get a contradiction, if $\omega_* \neq \omega^*$. Hence point of coincidence of F and f is unique. Now, suppose that ω_1^* is another point of coincidence of g and G, that is $\omega_1^* = G\omega_1 = g\omega_1$ for some $\omega_1 \in X$. Using (2.1), we have

$$\psi(\mathcal{P}(F\omega, G\omega_1, t), \mathcal{P}(f\omega, g\omega_1, t), \mathcal{P}(f\omega, F\omega, t),$$

$$\mathcal{P}(g\omega_1, G\omega_1, t), \mathcal{P}(f\omega, G\omega_1, t), \mathcal{P}(F\omega, g\omega_1, t)) \leq 0,$$

Thus,

$$\psi(\mathcal{P}(\omega^*, \omega_1^*, t), \mathcal{P}(\omega^*, \omega_1^*, t), \mathcal{P}(\omega^*, \omega^*, t),$$

$$\mathcal{P}(\omega_1^*, \omega_1^*, t), \mathcal{P}(\omega^*, \omega_1^*, t), \mathcal{P}(\omega^*, \omega_1^*, t)) \leq 0,$$

That is,

$$\psi(\mathcal{P}(\omega^*, \omega_1^*, t), \mathcal{P}(\omega^*, \omega_1^*, t), 0, 0, \mathcal{P}(\omega^*, \omega_1^*, t), \mathcal{P}(\omega^*, \omega_1^*, t)) \leq 0, \tag{2.24}$$

which contradicts (ψ_3), if $\omega_* \neq \omega^*$. Hence point of coincidence of G and g is unique. Then, we proved that ω^* is the unique point of coincidence of (F, f) and (G, g). Now, if (F, f) and (G, g) are weakly compatible, from $F\omega = f\omega = \omega^*$ and $G\omega' = g\omega' = \omega^*$, we have $F\omega^* = F(f\omega) = f(F\omega) = f\omega^*$ and $G\omega^* = G(g\omega') = g(G\omega') = g\omega^*$. Now, we prove that $F\omega^* = f\omega^* = G\omega^* = g\omega^*$. If not, then $F\omega^* \neq G\omega^*$ and from (2.1), we have

$$\psi(\mathcal{P}(F\omega^*, G\omega^*, t), \mathcal{P}(f\omega^*, g\omega^*, t), \mathcal{P}(f\omega^*, F\omega^*, t),$$

$$\mathcal{P}(g\omega^*, G\omega^*, t), \mathcal{P}(f\omega^*, G\omega^*, t), \mathcal{P}(F\omega^*, g\omega^*, t)) \leq 0,$$

That is,

$$\psi(\mathcal{P}(F\omega^*, G\omega^*, t), \mathcal{P}(F\omega^*, G\omega^*, t), 0, 0, \mathcal{P}(F\omega^*, G\omega^*, t), \mathcal{P}(F\omega^*, G\omega^*, t)) \leq 0 \tag{2.25}$$

By property (ψ_3), we deduce that $\mathcal{P}(F\omega^*, G\omega^*, t) \leq 0$ that is $\mathcal{P}(F\omega^*, G\omega^*, t) = 0$ and then our assumption that $F\omega^* \neq G\omega^*$ was wrong. Hence $F\omega^* = f\omega^* = G\omega^* = g\omega^*$. Finally, we show that $F\omega^* = f\omega^* = G\omega^* = g\omega^* = \omega^*$. Again, from (2.1) and using $F\omega^* = f\omega^* = G\omega^* = g\omega^*$, we obtain that

$$\psi(\mathcal{P}(F\omega, G\omega^*, t), \mathcal{P}(f\omega, g\omega^*, t), \mathcal{P}(f\omega, F\omega, t),$$

$$\mathcal{P}(g\omega^*, G\omega^*, t), \mathcal{P}(f\omega, G\omega^*, t), \mathcal{P}(F\omega, g\omega^*, t)) \leq 0,$$

That is,

$$\psi(\mathcal{P}(\omega^*, G\omega^*, t), \mathcal{P}(\omega^*, G\omega^*, t), 0,0, \mathcal{P}(\omega^*, G\omega^*, t), \mathcal{P}(\omega^*, G\omega^*, t)) \leq 0, \tag{2.26}$$

yielding thereby (due to (ψ_3)), $\mathcal{P}(\omega^*, G\omega^*, t) \leq 0$ and so $\mathcal{P}(\omega^*, G\omega^*, t) = 0$, a contradiction if $\omega^* \neq G\omega^*$. Hence $F\omega^* = f\omega^* = G\omega^* = g\omega^* = \omega^*$. Then ω^* is the unique common fixed point of F, f, g and G. The proof for the case in which $\overline{F(X)}$ is complete is similar and is therefore omitted. This completes the proof.

For mapping $G: X \to X$, we denote $\mathfrak{F}(G) = \{x \in X: x = Gx\}$.

Theorem 2.2: Let F, G, f and g be four self-maps of a parametric metric space $(X, \mathcal{P})$ satisfying the conditions (2.1) for all $x, y \in X$ and $t > 0$, then

$$\mathfrak{F}(F) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g) = \mathfrak{F}(G) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g) \tag{2.27}$$

Proof: Let $\omega^* \in \mathcal{F}(F) \cap \mathcal{F}(f) \cap \mathcal{F}(g)$. Then using (2.1), we have

$$\psi(\mathcal{P}(F\omega^*, G\omega^*, t), \mathcal{P}(f\omega^*, g\omega^*, t), \mathcal{P}(f\omega^*, F\omega^*, t),$$

$$\mathcal{P}(g\omega^*, G\omega^*, t), \mathcal{P}(f\omega^*, G\omega^*, t), \mathcal{P}(F\omega^*, g\omega^*, t)) \leq 0,$$

That is,

$$\psi(\mathcal{P}(\omega^*, G\omega^*, t), 0,0, \mathcal{P}(\omega^*, G\omega^*, t), \mathcal{P}(\omega^*, G\omega^*, t), 0) \leq 0,$$

By property (ψ_{2a}), we deduce that $\mathcal{P}(\omega^*, G\omega^*, t) \leq 0$ and so $\mathcal{P}(\omega^*, G\omega^*, t) = 0$, a contradiction if $\mathcal{P}(\omega^*, G\omega^*, t) > \mathcal{P}$. This means that

$$\omega^* \in \mathfrak{F}(G) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g)$$

Thus,

$$\mathfrak{F}(F) \cap \mathfrak{F}(f) \cap \mathfrak{F}(\mathrm{J}) \subset \mathfrak{F}(G) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g).$$

Similarly, we can show that

$$\mathfrak{F}(G) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g) \subset \mathfrak{F}(F) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g).$$

Thus, it follows that

$$\mathfrak{F}(G) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g) = \mathfrak{F}(F) \cap \mathfrak{F}(f) \cap \mathfrak{F}(g).$$

From Theorem 2.1, we can deduce a host of corollaries which are embodied in the following:

Corollary 2.3: The conclusions of Theorem 2.1 remain true if for all $x, y \in X; (x \neq y)$ and $t > 0$, the implicit relation (2.1) is replaced by one of the following:

(1) $\mathcal{P}(Fx,Gy,t) \leq k\ max\left\{\mathcal{P}(fx,\ gy,t),\mathcal{P}(fx,\ Fx,t),\mathcal{P}(gy,Gy,t),\frac{1}{2}[\mathcal{P}(fx,Gy,t)+\mathcal{P}(Fx,gy,t)]\right\}$

where $k \in (0,1)$.

(2) $\mathcal{P}(Fx,Gy,t) \leq k\ max\left\{\mathcal{P}(fx,\ gy,t),\mathcal{P}(fx,\ Fx,t),\mathcal{P}(gy,Gy,t),\frac{1}{2}\mathcal{P}(fx,Gy,t),\frac{1}{2}\mathcal{P}(Fx,gy,t)\right\}$

where $k \in (0,1)$.

(3) $\mathcal{P}(Fx,Gy,t) \leq k \max\left\{\mathcal{P}(fx,\ gy,t),\frac{1}{2}[\mathcal{P}(fx,\ Fx,t)+\mathcal{P}(gy,Gy,t)],\frac{1}{2}[\mathcal{P}(fx,Gy,t)+\mathcal{P}(Fx,gy,t)]\right\}$

where $k \in (0,1)$

(4) $\mathcal{P}(Fx,Gy,t) \leq a\mathcal{P}(fx,gy,t)+b\mathcal{P}(fx,\ Fx,t)+c\mathcal{P}(gy,Gy,t)+d\mathcal{P}(fx,Gy,t)+e\mathcal{P}(Fx,gy,t)$

where $a+b+c+d+e<1,\ d,e \geq 0$.

(5) $\mathcal{P}(Fx,Gy,t) \leq \frac{k}{2}\ max\{\mathcal{P}(fx,\ gy,t),\mathcal{P}(fx,\ Fx,t),\mathcal{P}(gy,Gy,t),\mathcal{P}(fx,Gy,t),\mathcal{P}(Fx,gy,t)\}$

where $k \in (0,1)$.

(6) $\mathcal{P}(Fx,Gy,t) \leq a\mathcal{P}(fx,\ gy,t)+b\mathcal{P}(fx,\ Fx,t)+c\mathcal{P}(gy,Gy,t)+d[\mathcal{P}(fx,Gy,t)+\mathcal{P}(Fx,gy,t)]$

where $a+b+c+2d<1,\ d \geq 0$.

(7) $\mathcal{P}(Fx,Gy,t) \leq \phi\left(\max\left\{\mathcal{P}(fx,\ gy,t),\mathcal{P}(fx,\ Fx,t),\mathcal{P}(gy,Gy,t),\frac{1}{2}[\mathcal{P}(fx,Gy,t)+\mathcal{P}(Fx,gy,t)]\right\}\right)$

where $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing upper semi-continuous function with $\phi(0)=0$ and $\phi(v)<v$ for each $v>0$.

(8) $\mathcal{P}(Fx,Gy,t) \leq \phi(\mathcal{P}(fx,\ gy,t),\mathcal{P}(fx,\ Fx,t),\mathcal{P}(gy,Gy,t),\mathcal{P}(fx,Gy,t),\mathcal{P}(Fx,gy,t))$

where $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that with $\phi(v,v,av,bv,cv)<v$ for each $v>0$ and $a,b,c \geq 0$ with $a+b+c \leq 3$.

Setting $F=G$ and $f=g$ in Theorem 2.1, we get the following corresponding fixed-point theorem.

Corollary 2.4: Let F and g be two self-maps of a parametric metric space $(X,\mathcal{P})$ with $\overline{F(X)} \subseteq g(X)$ and for all $x,y \in X$, $t>0$ and some $\psi \in \Psi$,

$$\psi(\mathcal{P}(Fx,Fy,t),\mathcal{P}(gx,gy,t),\mathcal{P}(gx,Fx,t),$$
$$\mathcal{P}(gy,Fy,t),\mathcal{P}(gx,Fy,t),\mathcal{P}(Fx,gy,t)) \leq 0, \quad (2.28)$$

If $\overline{F(X)}$ is a complete subspace of X, then (F,g) has a unique point of coincidence in X. Moreover, if (F,g) is weakly compatible, then (F,g) has a unique common fixed point in X.

Remark 2.5: A corollary like Corollary 2.4 can be outlined in respect of Corollary 2.3 yielding thereby a host of fixed-point theorems.

Setting $g=f_X$ (the identity mapping on X) in Corollary 2.1, we get the following corresponding fixed-point theorem.

Corollary 2.6: Let F be a self-map of a parametric metric space $(X, \mathcal{P})$ such that for all $x, y \in X, t > 0$ and some $\psi \in \Psi$,

$$\psi(\mathcal{P}(Fx, Fy, t), \mathcal{P}(x, y, t), \mathcal{P}(x, Fx, t),$$

$$\mathcal{P}(y, Fy, t), \mathcal{P}(x, Fy, t), \mathcal{P}(Fx, y, t)) \leq 0, \tag{2.29}$$

If $\overline{F(X)}$ is a complete subspace of X, then S has a unique common fixed point in X.

Remark 2.7: A corollary like Corollary 2.6 can be outlined in respect of Corollary 2.3 yielding thereby a host of fixed-point theorems.

3 Application

As an application of Theorem 2.1, we prove a common fixed-point theorem for four finite families of mappings which runs as follows:

Theorem 3.1: Let $\{F_1, F_2, \ldots, F_m\}$, $\{G_1, G_2, \ldots, G_p\}$, $\{f_1, f_2, \ldots, f_q\}$ and $\{g_1, g_2, \ldots, g_r\}$ be four finite families of self-mappings of a parametric metric space $(X, \mathcal{P})$ with

$$F = \prod_{i=1}^{m} F_i \ , G = \prod_{j=1}^{p} G_j,$$

$$f = \prod_{k=1}^{q} f_k, \ g = \prod_{l=1}^{r} g_l.$$

satisfying condition (2.1) of Theorem 2.1. Suppose that $\overline{G(X)} \subseteq f(X)$ and $\overline{F(X)} \subseteq g(X)$, wherein one of $\overline{G(X)}$ and $\overline{F(X)}$ is a complete subspace of X, then (F, f) and (G, g) have a point of coincidence in X.

Moreover, if

$$F_o F_r = F_r F_o, \ f_u f_v = f_v f_u,$$

$$G_s G_t = G_t G_s, \qquad g_e g_h = g_h g_e,$$

$$F_o f_u = f_u F_o, \qquad G_s g_e = G_s g_e$$

for all $o, r \in \{1,2, \ldots, m\}$, $u, v \in \{1,2, \ldots, q\}$, $s, t \in \{1,2, \ldots, p\}$, and $e, h \in \{1,2, \ldots, r\}$, then for all $o \in \{1,2, \ldots, m\}, u \in \{1,2, \ldots, q\}, s \in \{1,2, \ldots, p\}$ and e $\in \{1,2, \ldots, r\}$, F_o, G_s, f_u and g_e have a common fixed point.

Proof: The conclusions " (F, f) and (G, g) have a point of coincidence in X " are immediate as F, G, f and g satisfy all the conditions of theorem 2.1. In view of pairwise commutativity of various pairs of the families (F, f) and (G, g), the weak compatibility of pairs (F, f) and (G, g) are immediate. Thus, all the conditions of theorem 2.1 (for mappings F, G, f and g) are satisfied ensuring the existence of a unique common fixed point, say ω^*. Now, one needs to show that ω^* remains the fixed point of all the component maps. For this consider

$$F(F_o\omega^*) = \left(\prod_{i=1}^{m} F_i\right)(F_o\omega^*)$$

$$= \left(\prod_{i=1}^{m-1} F_i\right)(F_m F_o)\omega^*$$

$$= \left(\prod_{i=1}^{m-1} F_i\right)(F_m F_o \omega^*)$$

$$= \left(\prod_{i=1}^{m-2} F_i\right)\left(F_{m-1} F_o (F_m \omega^*)\right)$$

$$= \left(\prod_{i=1}^{m-2} F_i\right)\left(F_o F_{m-1} (F_m \omega^*)\right)$$

$$= \cdots$$

$$= F_1 F_o \left(\prod_{i=2}^{m} F_i\, \omega^*\right)$$

$$= F_o F_1 \left(\prod_{i=2}^{m} F_i(\omega^*)\right)$$

$$= F_o \left(\prod_{i=1}^{m} F_i\, (\omega^*)\right)$$

$$= F_o(F\omega^*) = F_o\omega^* \tag{3.1}$$

Similarly, one can show that,

$$F(f_u\omega^*) = f_u(F\omega^*) = f_u\omega^*,$$

$$f(f_u\omega^*) = f_u(f\omega^*) = f_u\omega^*,$$

$$f(F_o\omega^*) = F_o(f\omega^*) = F_o\omega^*,$$

$$G(G_s\omega^*) = G_s(G\omega^*) = G_s\omega^*,$$

$$G(g_e\omega^*) = g_e(G\omega^*) = g_e\omega^*,$$

$$g(g_e\omega^*) = g_e(g\omega^*) = g_e\omega^*,$$

$$G(g_e\omega^*) = g_e(G\omega^*) = g_e\omega^*. \tag{3.2}$$

which show that (for all o $\in \{1,2,3, \dots, m\}$, $u \in \{1,2, \dots, q\}$, $s \in \{1,2, \dots, p\}$ and $e \in \{1,2, \dots, r\}$) $F_o\omega^*$ and $f_u\omega^*$ are other fixed points of the pair (F, f) whereas $G_s\omega^*$ and $g_e\omega^*$ are other fixed points of the pair (G, g).

Now in view of uniqueness of the fixed point F, G, f and g (for all $o \in \{1,2, \dots, m\}, u \in \{1,2, \dots, q\}, s \in \{1,2, \dots, p\}$ and $e \in \{1,2, \dots, r\}$), one can write $F_o\omega^* = f_u\omega^* = G_s\omega^* = g_e\omega^* = \omega^*$.

This means that the point ω^* is a common fixed point of F_o, f_u, G_s and g_e. for all $o \in \{1,2, \dots, m\}, u \in \{1,2, \dots, q\}, s \in \{1,2, \dots, p\}$ and $e \in \{1,2, \dots, r\}$. By setting

$$F_1 = F_2 = \cdots = F_m = F,$$

$$G_1 = G_2 = \cdots = G_p = G,$$

$$f_1 = f_2 = \cdots = f_q = f,$$

$$g_1 = g_2 = \cdots = g_r = g. \tag{3.3}$$

One deduces the following corollary for various iterates of F, G, f and g, which can also be viewed as partial generalization of theorem 2.1.

Corollary 3.2: Let (F, f) and (G, g) be two commuting pairs of self-mappings of a parametric metric space $(X, \mathcal{P})$ with $\overline{G^p(X)} \subseteq f^q(X)$ and $\overline{F^m(X)} \subseteq g^r(X)$ and for all $x, y \in X$, $t > 0$ and some $\psi \in \Psi$,

$$\psi(\mathcal{P}(F^m x, G^p y, t), \mathcal{P}(f^q x, g^r y, t), \mathcal{P}(f^q x, F^m x, t),$$

$$\mathcal{P}(g^r y, G^p y, t), \mathcal{P}(f^q x, G^p y, t), \mathcal{P}(F^m x, g^r y, t)\big) \leq 0, \tag{3.4}$$

If one of $\overline{G^p(X)}$ and $\overline{F^m(X)}$ is a complete subspace of X, then (F, f) and (G, g) have a unique point of coincidence in X. Moreover, if (F, f) and (G, g) are weakly compatible, then F, G, f and g have a unique common fixed point in X.

Theorem 3.3: Let $\{F_1, F_2, \ldots, F_m\}$ and $\{g_1, g_2, \ldots, g_r\}$ be two finite families of self-mappings of a parametric metric space $(X, \mathcal{P})$ with $F = \prod_{i=1}^{m} F_i$, $g = \prod_{j=1}^{r} g_j$ satisfying condition (2.28) of Corollary 2.4. Suppose that $\overline{F(X)} \subseteq g(X)$, wherein $\overline{F(X)}$ is a complete subspace of X, then (F, g) have a unique point of coincidence.

Moreover, if $F_p F_q = F_q F_p$, $g_k g_l = \mathrm{J}_l g_k$ and $F_i g_k = g_k F_i$ for all $p, q \in \{1,2, \ldots, m\}$ and $k, l \in \{1,2, \ldots, r\}$, then ($p \in \{1,2, \ldots, m\}$ and $k \in \{1,2, \ldots, p\}$) F_p and g_k have a common fixed point in X.

Proof: The conclusion "(F, g) has a point of coincidence" is immediate as F and g satisfies all the conditions of Corollary 2.4. Now appealing to component wise commutativity of various pairs, one can immediately assert that $Fg = gF$ and hence, obviously the pair (F, g) is weakly compatible. Note that all the conditions (2.28) of Corollary 2.4 (for mappings F and g) are satisfied ensuring the existence of unique common fixed point, say ω^*. Now one need to show that ω^* remains the fixed point of all the component mappings. For this consider

$$\begin{aligned}
F\big(F_p \omega^*\big) &= \big(\textstyle\prod_{i=1}^{m} F_i\big)\big(F_p \omega^*\big) \\
&= \big(\textstyle\prod_{i=1}^{m-1} F_i\big)(F_m F_p)\omega^* \\
&= \big(\textstyle\prod_{i=1}^{m-1} F_i\big)\big(F_m F_p \omega^*\big) \\
&= \big(\textstyle\prod_{i=1}^{m-2} F_i\big)\Big(F_{m-1} F_p (F_m \omega^*)\Big) \\
&= \big(\textstyle\prod_{i=1}^{m-2} F_i\big)\Big(F_p F_{m-1} (F_m \omega^*)\Big) \\
&= \cdots \\
&= F_1 F_p \big(\textstyle\prod_{i=2}^{m} F_i \, \omega^*\big) \\
&= F_p F_1 \big(\textstyle\prod_{i=2}^{m} F_i (\omega^*)\big) \\
&= F_p\big(\textstyle\prod_{i=1}^{m} F_i \, (\omega^*)\big) \\
&= F_p(F\omega^*) = F_p \omega^*
\end{aligned} \tag{3.5}$$

Similarly, one can show that,

$$F(g_k\omega^*) = g_k(F\omega^*) = g_k\omega^*,$$

$$g(g_k\omega^*) = g_k(g\omega^*) = g_k\omega^*,$$

$$g(F_p\omega^*) = F_p(g\omega^*) = g_p\omega^*, \tag{3.6}$$

which show that (for all $p \in \{1,2,\ldots,m\}, k \in \{1,2,\ldots,r\}$) $F_p\omega^*$ and $g_k\omega^*$ are other fixed points of the pair (F, g).

Now in view of uniqueness of the fixed point F, G, f and g (for all $p \in \{1,2,\ldots,m\}$, $k \in \{1,2,\ldots,r\}$), one can write $F_p\omega^* = g_k\omega^* = \omega^*$.

This means that the point ω^* is a common fixed point of F_p and g_k. for all $p \in \{1,2,\ldots,m\}$, $k \in \{1,2,\ldots,r\}$. By setting

$$F_1 = F_2 = \cdots = F_m = F,$$

$$g_1 = g_2 = \cdots = g_r = g. \tag{3.7}$$

One deduces the following corollary for various iterates of F and g, which can also be viewed as partial generalization of Corollary 2.1.

Corollary 3.4: Let (F, g) be two commuting pairs of self-mappings of a parametric metric space $(X, \mathcal{P})$ with $\overline{F^m(X)} \subseteq g^r(X)$ and for all $x, y \in X$, $t > 0$ and some $\psi \in \Psi$,

$$\psi(\mathcal{P}(F^m x, F^m y, t), \mathcal{P}(g^r x, g^r y, t), \mathcal{P}(g^r x, F^m x, t),$$

$$\mathcal{P}(g^r y, F^m y, t), \mathcal{P}(g^r x, F^m y, t), \mathcal{P}(F^m x, g^r y, t)) \leq 0, \tag{3.8}$$

Assume that $\overline{F^m(X)}$ is a complete subspace of X, then (F, g) has a unique point of coincidence in X. Moreover, if (F, g) is weakly compatible, then (F, g) has a unique common fixed point in X.

4 Example

Now we furnish an example to demonstrate the validity of the hypotheses of generality of our result.

Example 4.1: Let $X = \{0,1,3,4\}$ be endowed with parametric metric $\mathcal{P}(x, y, t) = t|x - y|$ for all $x, y \in X$ and all $t > 0$. Then $(X, \mathcal{P})$ is a parametric metric space.

Also define the mappings $F, G, f, g: X \to X$ by

$$Fx = 1, \ \forall\, x \in X,$$

$$Gx = \begin{cases} 0, & x \in \{3\} \\ 1, & x \in \{0,1,2\}. \end{cases}$$

and

$$fx = gx = x, \qquad \forall\, x \in X$$

that is, $f = g = f_X$ (the identity mapping on X). We can see that the mappings (F, f) and (G, J) are commute at 1 which is their coincidence point. Obviously, (F, f) and (G, g) are weakly compatible.

Also $F(X) = \{1\}, G(X) = \{0,1\}$ and $f(X) = F(X) = \{0,1,3,4\}$. Clearly, $\overline{F(X)} = \{1\} \subset \{0,1,3,4\} = g(X)$ and $\overline{G(X)} = \{0,1\} \subset \{0,1,3,4\} = f(X)$ are complete subspace of X.

Now, we define $\psi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6): \mathbb{R}_+^6 \to \mathbb{R}$ as:

$$\psi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = \xi_1 - a_1\left(\frac{\xi_3^2 + \xi_4^2}{\xi_3 + \xi_4}\right) - a_2\xi_2 - a_3(\xi_5 + \xi_6)$$

where $a_i \geq 0$ with at least one a_i non-zero and $a_1 + a_2 + 2a_3 < 1$. Now taking $a_1 = \frac{1}{5}, a_2 = a_3 = \frac{1}{4}$, we consider the following cases.

(1). Let $x = 0, y = 1$ and for all $\mathrm{t} > 0$. Then

$$\psi(\mathcal{P}(F0, G1, t), \mathcal{P}(f1, g0, t), \mathcal{P}(f0, F0, t), \mathcal{P}(g1, G1, t), \mathcal{P}(f0, G1, t), \mathcal{P}(F0, g1, t))$$

$$= \psi(\mathcal{P}(1,1, t), \mathcal{P}(1, 0, t), \mathcal{P}(0, 1, t), \mathcal{P}(1,1, t), \mathcal{P}(0,1, t), \mathcal{P}(1,1, t))$$

$$= \psi(0, \mathrm{t}, \mathrm{t}, 0, \mathrm{t}, \mathrm{t})$$

$$= 0 - a_1\left(\frac{\mathrm{t}+0}{\mathrm{t}+0}\right) - a_2\mathrm{t} - a_3(\mathrm{t} + \mathrm{t})$$

$$= -\frac{1}{5} + \frac{-3}{4}\mathrm{t} < 0.$$

(2). Let $x = 0, y = 3$ and for all $\mathrm{t} > 0$. Then

$$\psi(\mathcal{P}(F0, G3, t), \mathcal{P}(f0, g3, t), \mathcal{P}(f0, F0, t), \mathcal{P}(g3, G3, t), \mathcal{P}(f0, G3, t), \mathcal{P}(F0, g3, t))$$

$$= \psi(\mathcal{P}(1,0, t), \mathcal{P}(0, 3, t), \mathcal{P}(0, 1, t), \mathcal{P}(3,0, t), \mathcal{P}(0,0, t), \mathcal{P}(1,3, t))$$

$$= \psi(\mathrm{t}, 3\mathrm{t}, \mathrm{t}, 3\mathrm{t}, 0, 2\mathrm{t})$$

$$= \mathrm{t} - a_1\left(\frac{\mathrm{t}+9\mathrm{t}^2}{\mathrm{t}+3}\right) - 3a_2 - a_2(0 + 2\mathrm{t})$$

$$= \mathrm{t} - \frac{1}{5}\left(\frac{\mathrm{t}+9\mathrm{t}^2}{\mathrm{t}+3}\right) - \frac{3\mathrm{t}}{4} - \frac{1}{4}(0 + 2\mathrm{t})$$

$$= -\frac{1}{4}\mathrm{t} - \frac{1}{5}\left(\frac{\mathrm{t}+9\mathrm{t}^2}{\mathrm{t}+3}\right) < 0$$

$$= -\frac{1}{4}\mathrm{t} - \frac{1}{5}\left(\frac{\mathrm{t}+9\mathrm{t}^2}{\mathrm{t}+3}\right) < 0$$

(3). Let $x = 0, y = 4$ and for all $\mathrm{t} > 0$. Then

$$\psi(\mathcal{P}(F0, G4, t), \mathcal{P}(f0, g4, t), \mathcal{P}(f0, F0, t), \mathcal{P}(g4, G4, t), \mathcal{P}(f0, G4, t), \mathcal{P}(F0, g4, t))$$

$$= \psi(\mathcal{P}(1,1, t), \mathcal{P}(0, 4, t), \mathcal{P}(0, 1, t), \mathcal{P}(4,1, t), \mathcal{P}(0,1, t), \mathcal{P}(1,4, t))$$

$= \psi(0{,}4\mathrm{t}, \mathrm{t}, 3\mathrm{t}, \mathrm{t}, 3\mathrm{t})$

$= 0 - a_1\left(\frac{\mathrm{t}+9\mathrm{t}^2}{\mathrm{t}+3}\right) - 4\mathrm{t}a_2 - a_3(\mathrm{t}+3\mathrm{t})$

$= 0 - \frac{1}{5}\left(\frac{\mathrm{t}+9\mathrm{t}^2}{\mathrm{t}+3}\right) - \mathrm{t} - \mathrm{t}$

$= -2\mathrm{t} - \frac{1}{5}\left(\frac{\mathrm{t}+9\mathrm{t}^2}{\mathrm{t}+3}\right) < 0.$

(4). Let $x = 1, y = 3$ and for all $\mathrm{t} > 0$. Then,

$\psi(\mathcal{P}(F1, G3, t), \mathcal{P}(f1, g3, t), \mathcal{P}(f1, F1, t), \mathcal{P}(g3, G3, t), \mathcal{P}(f1, G3, t), \mathcal{P}(F1, g3, t))$

$= \psi(\mathcal{P}(1{,}0, t), \mathcal{P}(1, 3, t), \mathcal{P}(1, 1, t), \mathcal{P}(3{,}0, t), \mathcal{P}(1{,}0, t), \mathcal{P}(1{,}3, t))$

$= \psi(\mathrm{t}, 2\mathrm{t}, 0{,}3\mathrm{t}, \mathrm{t}, 2)$

$= \mathrm{t} - a_1\left(\frac{0+9\mathrm{t}^2}{0+3\mathrm{t}}\right) - a_2 2\mathrm{t} - a_3(\mathrm{t}+2\mathrm{t})$

$= \mathrm{t} - \frac{3}{5}\mathrm{t} - \frac{1}{2}\mathrm{t} - \frac{3}{4}\mathrm{t}$

$= \frac{-17}{20}\mathrm{t} < 0.$

(5). Let $x = 1, y = 4$ and for all $\mathrm{t} > 0$. Then,

$\psi(\mathcal{P}(F1, G4, t), \mathcal{P}(f1, g4, t), \mathcal{P}(f1, F1, t), \mathcal{P}(g4, G4, t), \mathcal{P}(f1G4, t), \mathcal{P}(F1, g4, t))$

$= \psi(\mathcal{P}(1{,}1, t), \mathcal{P}(1, 4, t), \mathcal{P}(1, 1, t), \mathcal{P}(4{,}1, t), \mathcal{P}(1{,}1, t), \mathcal{P}(1{,}4, t))$

$= \psi(0{,}3\mathrm{t}, 0{,}3\mathrm{t}, 0{,}3\mathrm{t})$

$= 0 - a_1\left(\frac{0+9\mathrm{t}^2}{0+3\mathrm{t}}\right) - 3\mathrm{t}a_2 - a_3(0+3\mathrm{t})$

$= -\frac{3}{5}\mathrm{t} - \frac{3}{4}\mathrm{t} - \frac{3}{4}\mathrm{t}$

$= \frac{-42}{20}\mathrm{t} < 0.$

(6). Let $x = 3, y = 4$ and for all $\mathrm{t} > 0$. Then,

$\psi(\mathcal{P}(F3, G4, t), \mathcal{P}(f3, g4, t), \mathcal{P}(f3, F3, t), \mathcal{P}(g4, G4, t), \mathcal{P}(f3, G4, t), \mathcal{P}(F3, g4, t))$

$= \psi(\mathcal{P}(1{,}1, t), \mathcal{P}(3{,}4, t), \mathcal{P}(3, 1, t), \mathcal{P}(4{,}1, t), \mathcal{P}(3{,}1, t), \mathcal{P}(3{,}4, t))$

$= \psi(0, \mathrm{t}, 3\mathrm{t}, 4\mathrm{t}, 3\mathrm{t}, \mathrm{t})$

$= 0 - a_1\left(\frac{9\mathrm{t}^2+16\mathrm{t}^2}{3\mathrm{t}+4\mathrm{t}}\right) - ta_2 - \mathrm{a}_3(3\mathrm{t}+\mathrm{t})$

$= -\frac{5}{7}\mathrm{t} - \frac{1}{4}\mathrm{t} - \mathrm{t}$

$= \frac{-55}{28}\mathrm{t} < 0.$

Therefore, all condition of Theorem 2.1 hold and S, T , I and J have a unique common fixed point $(\omega^* = 1)$.

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Competing Interests

Authors have declared that no competing interests exist.

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