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On Generalized Opial's Integral Inequalities in q-Calculus

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Authors' contributions

This work was carried out in collaboration among the authors. All the authors read and approved the final manuscript.

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Abstract

In this paper, we establish results for q-analogues of generalized Opial integral inequalities and also present some extensions of the analogues. Using the concepts of q-differentiability and continuity of functions and the application of the Hölder's integral inequality we establish the results.

Keywords: Generalized Opial Inequality; Hölder's Integral Inequality; q-analogue; q-Calculus.

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1 Introduction

Opial established an inequality involving integral of a function and its derivative ([1]) as

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4}\int_0^h (f'(x))^2 dx, \tag{1.1}$$

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where $f \in C^1[0, h]$, such that $f(0) = f(h) = 0$, $f'(x) > 0$ and $x \in [0, h]$. The coefficient $h/4$ is the best constant possible.

This inequality, due to its significance, experienced a lot of extensions and generalizations over time in both classical and q-analogues. In [2], generalizations of the classical Opial's inequality were established as

$$\int_a^b \left|f(x)f'(x)\right| dx \leq \frac{(b-a)}{2} \int_a^b \left|f'(x)\right|^2 dx \tag{1.2}$$

and

$$\int_a^b \left|f(x)f'(x)\right| dx \leq \frac{(b-a)}{4} \int_a^b \left|f'(x)\right|^2 dx, \tag{1.3}$$

where the coefficients $(b-a)/2$ and $(b-a)/4$ are their respective best constants possible.

In [3], the authors established a q-analogue of a generalized Opial type inequality as

$$\int_a^b |D_q f(x)||f(x)|^p d_q x \leq (b-a)^p \int_a^b |D_q f(x)|^{p+1} d_q x, \tag{1.4}$$

where $f \in C^1[a, b]$ is a q-decreasing function with $f(bq^0) = 0$ and $p \geq 0$.

See also ([4], [5], [6] and [7]) for more q-analogues of the Opial's type inequalities. q-Calculus is a mathematical field of study which is analogous to the ordinary calculus. It is used to find q-derivatives and q-integrals of functions ([8]).

The Opial inequality plays essential role in establishing the existence and uniqueness of initial and boundary values problems for both ordinary and partial differential equations as well as in difference equations ([4] and [8]).

Motivated by q-calculus our objective in this paper is to establish q-analogues of the generalized Opial Integral Inequalities (1.2) and (1.3).

2 Preliminaries

In this section, the basic concepts and terminologies of q-calculus are presented. The definitions provided can also be seen in ([9], [10], [11], [12], [13], [8], [14], [15], [7] and [16]).

Definition 2.1. For any arbitrary function f, the q-derivative $(D_q f)$ is defined as

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0. \tag{2.1}$$

Notation 2.1. *For any positive real α, the q-number*

$$[\alpha]_q = \frac{1-q^\alpha}{1-q} = 1 + q + q^2 + \cdots + q^{\alpha-1},$$

for $0 < q < 1$, $\alpha \in \mathbf{R}^+$.

Definition 2.2. The q-Derivative of product of f and g is defined as

$$\begin{aligned} D_q(f(x)g(x)) &= f(x)D_q g(x) + g(qx)D_q f(x) \\ &= f(qx)D_q g(x) + g(x)D_q f(x). \end{aligned} \tag{2.2}$$

Definition 2.3. (Composite Rule) Let f be a function of a power function g, the q-derivative is defined as

$$D_q f(g(x)) = D_{q^k}(f(g(x)))\, D_q(g)(x), \tag{2.3}$$

where k is a real and index of g ([8]).

Lemma 2.2. [17] *For any positive real* α*, then we have*

$$D_q(x-a)^\alpha = [\alpha]_q (x-a)^{\alpha-1}, \tag{2.4}$$

for $0 < q < 1$, $\alpha \in \mathbf{R}^+$.

Proof.

$$\begin{aligned} D_q(x-a)^\alpha &= \frac{(x-a)^\alpha - ((x-a)q)^\alpha}{(1-q)(x-a)} \\ &= [\alpha]_q (x-a)^{\alpha-1}. \end{aligned}$$

This completes the proof of the lemma. □

Definition 2.4. Let $f : C[0,b] \to \mathbb{R}$ $(b > 0)$. Then the Jackson's definite q-Integral on $[0,b]$ is defined as

$$\int_0^b f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} q^j f(bq^j). \tag{2.5}$$

The q-integral on [a, b] is defined as

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.6}$$

Definition 2.5. The function f defined on $[a,b]$ is called q-increasing (q-decreasing) on $[a,b]$, if $f(qx) \le f(x) \quad (f(qx) \ge f(x))$, for x, $qx \in [a,b]$ ([11]).

It is easily observed that if the function f is increasing (decreasing), then it is also q-increasing (q-decreasing).

Definition 2.6. A function $f : \mathbf{I} \to \mathbf{R}$ is said to be convex on I if for every x, $y \in \mathbf{I}$ and $0 \le \lambda \le 1$ the inequality

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) \tag{2.7}$$

holds.

3 Main Results

Lemma 3.1. *Let* $h : [a,b] \to \mathbf{R}$ *be a differentiable function, such that* $D_q h \in L_p[a,b]$, $1 \le p < \infty$ *and* $0 < q < 1$*. Then,*

$$\left(\int_a^b |D_q h(x)| d_q x \right)^p \le (b-a)^{p-1} \int_a^b |D_q h(x)|^p d_q x. \tag{3.1}$$

Proof. Applying Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}\left(\int_a^b |D_qh(x)|d_qx\right)^p &= \left(\int_a^b t^{\frac{1}{p}}|D_qh(x)|t^{-\frac{1}{p}}d_qx\right)^p \\ &\le \left[\left(\int_a^b t|D_qh(x)|^p d_qx\right)^{\frac{1}{p}}\left(\int_a^b (t^{-\frac{1}{p}})^{\frac{p}{p-1}}d_qx\right)^{\frac{p-1}{p}}\right]^p \\ &= \int_a^b t|D_qh(x)|^p d_qx\left(\int_a^b t^{-\frac{1}{p-1}}d_qx\right)^{p-1} \\ &= (b-a)^{p-1}\int_a^b |D_qh(x)|^p d_qx.\end{aligned}$$

This completes the proof of the lemma. □

Theorem 3.2. *Let $h:[a,b]\to \mathbf{R}$ be a differentiable function such that $D_qh\in L_p[a,b]$, $h(a)=0,$ (or $\quad h(b)=0$), $0<q<1$ and $1\le p<\infty$. Then*

$$\int_a^b |D_qh(x)||h(x)|^{p-1}d_qx \le \frac{(b-a)^{p-1}}{[p]_q}\int_a^b |D_qh(x)|^p d_qx \tag{3.2}$$

holds.

Proof. Let ϕ be a convex function on $[0,\infty)$ with $\phi(0)=0$, $x\in[a,b]$, $h(a)=0$ and

$$y(x)=\int_a^x |D_qh(t)|d_qt.$$

Also, let

$$J(x)=\phi(y(x))=\phi\left(\int_a^x |D_qh(t)|d_qt\right). \tag{3.3}$$

Since $D_qy(x)=|D_qh(x)|$ and $|h(x)|\le y(x)$, then

$$D_qJ(x)=D_q\phi(y(x))|D_qh(x)|\ge D_q\phi(|h(x)|)|D_qh(x)|. \tag{3.4}$$

Thus

$$\int_a^b D_qJ(x)d_qx=\phi\left(y\left(b\right)\right)-\phi(y(a))\ge \int_a^b D_q\phi(|h(x)|)|D_qh(x)|d_qx. \tag{3.5}$$

Since $\phi(0)=0$, (3.5) becomes

$$\int_a^b D_q\phi(|h(x)|)|D_qh(x)|d_qx\le \phi\left(\int_a^b |D_qh(x)|d_qx\right). \tag{3.6}$$

Considering $\phi(x)=\frac{x^p}{p}$ for $1\le p<\infty$ in (3.6) we obtain

$$\frac{[p]_q}{p}\int_a^b |D_qh(x)||h(x)|^{p-1}d_qx\le \frac{1}{p}\left(\int_a^b |D_qh(x)|d_qx\right)^p. \tag{3.7}$$

Applying Lemma 3.1 to the right-hand side of (3.7) yields

$$\frac{[p]_q}{p}\int_a^b |D_qh(x)||h(x)|^{p-1}d_qx\le \frac{(b-a)^{p-1}}{p}\int_a^b |D_qh(x)|^p d_qx, \tag{3.8}$$

which implies that

$$\int_a^b |D_qh(x)||h(x)|^{p-1}d_qx\le \frac{(b-a)^{p-1}}{[p]_q}\int_a^b |D_qh(x)|^p d_qx.$$

This completes the proof of the theorem. □

Remark 3.1. Taking $p = 2$ in (3.2) yields

$$\int_a^b |D_q h(x)||h(x)|d_q x \leq \frac{(b-a)}{[2]_q} \int_a^b |D_q h(x)|^2 d_q x. \tag{3.9}$$

This simplifies to

$$\begin{aligned} \int_a^b |D_q h(x)||h(x)|d_q x &\leq \frac{(1-q)(b-a)}{(1-q^2)} \int_a^b |D_q h(x)|^2 d_q x \\ &= \frac{(b-a)}{1+q} \int_a^b |D_q h(x)|^2 d_q x. \end{aligned} \tag{3.10}$$

which is the q-analogue of (1.2).

Remark 3.2. By taking the limit as $q \to 1^-$ in (3.10) yields (1.2).

Theorem 3.3. *Let $h \in C^n[a,b]$ be a differentiable function, such that $D_q^{(i)} h(a) = 0$, for $i = 1, 2, \ldots n-1$, $1 \leq p < \infty$ and $0 < q < 1$. Then*

$$\int_a^b (x-a)^{n-1} |D_q^n h(x)||h(x)|^{p-1} d_q x \leq \frac{(b-a)^{pn-1}}{[p]_q} \int_a^b |D_q h(x)|^p d_q x \tag{3.11}$$

holds.

Proof. Let ϕ be a convex function on $[0, \infty)$ with $\phi(0) = 0$, $x \in [a,b]$, $D_q^{(i)} h(a) = 0$ and

$$y(x) = \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} |D_q h(s)| d_q s d_q t_1 \ldots d_q t_{n-1},$$

so that
$D_q^{(i)} y(x) \geq 0, \quad D_q^{(n)} y(x) = |D_q^{(n)} h(x)|, \quad \text{and} \quad y(x) \geq |h(x)|.$

By the mean value theorem for integral, it follows that

$$D_q^{(i)} y(x) \leq (x-a) D_q^{(i+1)} y(x), \quad x \in [a,b], \quad 0 \leq i \leq n-2. \tag{3.12}$$

It implies that

$$|h(x)| \leq y(x) \leq (x-a) D_q y(x) \leq \cdots \leq (x-a)^{n-1} D_q^{n-1} y(x).$$

Consider

$$F(x) = \phi((x-a)^{n-1} D_q^{n-1} y(x)). \tag{3.13}$$

Applying Lemma 2.2, then

$$\begin{aligned} D_q F(x) = D_q \phi((x-a)^{n-1} D_q^{n-1} y(x))[[n-1]_q (x-a)^{n-2} D_q^{n-1} y(x) \\ + (x-a)^{n-1} D_q^n y(x)]. \end{aligned} \tag{3.14}$$

From (3.14) we obtain

$$\begin{aligned} D_q F(x) &\geq D_q \phi(|h(x)|)(x-a)^{n-1} D_q^n y(x) \\ &= D_q \phi(|h(x)|)(x-a)^{n-1} |D_q^n h(x)|. \end{aligned} \tag{3.15}$$

Thus

$$\begin{aligned} \int_a^b D_q F(x) d_q x &= \phi((b-a)^{n-1} D_q^{n-1} y(b)) - \phi(0) \\ &\geq \int_a^b D_q \phi(|h(x)|)(x-a)^{n-1} |D_q^n h(x)| d_q x. \end{aligned} \tag{3.16}$$

Since $\phi(0) = 0$, (3.16) becomes

$$\int_a^b D_q\phi(|h(x)|)(x-a)^{n-1}|D_q^n h(x)|d_qx \leq \phi\left((b-a)^{n-1}\int_a^b D_q^n y(x)d_qx\right), \quad (3.17)$$

which implies that

$$\int_a^b D_q\phi(|h(x)|)(x-a)^{n-1}|D_q^n h(x)|d_qx \leq \phi\left((b-a)^{n-1}\int_a^b |D_q^n h(x)|d_qx\right). \quad (3.18)$$

Considering $\phi(x) = \frac{x^p}{p}$ for $1 \leq p < \infty$ in (3.18) we obtain

$$\frac{[p]_q}{p}\int_a^b (x-a)^{n-1}|D_q^n h(x)||h(x)|^{p-1}d_qx \leq \frac{1}{p}\left((b-a)^{n-1}\int_a^b |D_q^n h(x)|d_qx\right)^p. \quad (3.19)$$

This simplifies to

$$\frac{[p]_q}{p}\int_a^b (x-a)^{n-1}|D_q^n h(x)||h(x)|^{p-1}d_qx \leq \frac{(b-a)^{p(n-1)}}{p}\left(\int_a^b |D_q^n h(x)|d_qx\right)^p. \quad (3.20)$$

Applying Lemma 3.1 into the right-hand side of (3.20) yields

$$\frac{[p]_q}{p}\int_a^b (x-a)^{n-1}|D_q^n h(x)||h(x)|^{p-1}d_qx \leq \frac{(b-a)^{p(n-1)}(b-a)^{p-1}}{p}\int_a^b |D_q^n h(x)|^p d_qx, \quad (3.21)$$

which implies that

$$\int_a^b (x-a)^{n-1}|D_q^n h(x)||h(x)|^{p-1}d_qx \leq \frac{(b-a)^{pn-1}}{[p]_q}\int_a^b |D_q h(x)|^p d_qx.$$

This completes the proof of the theorem. □

Remark 3.3. Taking $p = 2$ in (3.11) yields

$$\int_a^b (x-a)^{n-1}|D_q^n h(x)||h(x)|d_qx \leq \frac{(b-a)^{2n-1}}{[2]_q}\int_a^b |D_q h(x)|^2 d_qx. \quad (3.22)$$

This simplifies to

$$\int_a^b (x-a)^{n-1}|D_q^n h(x)||h(x)|d_qx \leq \frac{(1-q)(b-a)^{2n-1}}{(1-q^2)}\int_a^b |D_q h(x)|^2 d_qx = \frac{(b-a)^{2n-1}}{1+q}\int_a^b |D_q h(x)|^2 d_qx, \quad (3.23)$$

for $n \geq 1$.

Remark 3.4. For $n = 1$ and by taking the limit as $q \to 1^-$ in (3.23) yields (1.2).

Theorem 3.4. *Let $h : [a, b] \to \mathbf{R}$ be a differentiable function such that $D_qh \in L_p[a, b]$, $h(a) = h(b) = 0$, $1 \leq p < \infty$, and $0 < q < 1$. Then*

$$\int_a^b |D_q h(x)||h(x)|^{p-1}d_qx \leq \frac{(b-a)^{p-1}}{2^{p-1}[p]_q}\int_a^b |D_q h(x)|^p d_qx \quad (3.24)$$

holds.

Proof. Let ϕ be a convex function on $[0, \infty)$ with $\phi(0) = 0$, $x \in [a, b]$, $h(a) = 0$ and

$$y(x) = \int_a^x |D_q h(t)| d_q t,$$

so that

$$J(x) = \phi(y(x)) = \phi\left(\int_a^x |D_q h(t)| d_q t\right). \tag{3.25}$$

Since $D_q y(x) = |D_q h(x)|$ and $|h(x)| \leq y(x)$, then

$$D_q J(x) = D_q \phi(y(x)) |D_q h(x)| \geq D_q \phi(|h(x)|) |D_q h(x)|. \tag{3.26}$$

Also, let

$$z(x) = \int_x^b |D_q h(t)| d_q t \tag{3.27}$$

for $h(b) = 0$, then

$$T(x) = -\phi(z(x)) = -\phi\left(\int_x^b |D_q h(t)| d_q t\right). \tag{3.28}$$

Since $D_q z(x) = -|D_q h(x)|$ and $|h(x)| \leq z(x)$, then

$$D_q T(x) = D_q \phi(z(x)) |D_q h(x)| \geq D_q \phi(|h(x)|) |D_q h(x)|. \tag{3.29}$$

Let $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ be subintervals of $[a, b]$.
By (3.26) we obtain

$$\begin{aligned} \int_a^{\frac{a+b}{2}} D_q J(x) d_q x &= \phi\left(y\left(\frac{a+b}{2}\right)\right) - \phi(y(a)) \\ &\geq \int_a^{\frac{a+b}{2}} D_q \phi(|h(x)|) |D_q h(x)| d_q x. \end{aligned} \tag{3.30}$$

Since $\phi(0) = 0$, thus

$$\phi\left(\int_a^{\frac{a+b}{2}} |D_q h(x)| d_q x\right) \geq \int_a^{\frac{a+b}{2}} D_q \phi(|h(x)|) |D_q h(x)| d_q x. \tag{3.31}$$

Also, by (3.29) we obtain

$$\begin{aligned} \int_{\frac{a+b}{2}}^b D_q T(x) d_q x &= \phi(z(b)) - \phi\left(z\left(\frac{a+b}{2}\right)\right) \\ &\geq \int_{\frac{a+b}{2}}^b D_q \phi(|h(x)|) |D_q h(x)| d_q x. \end{aligned} \tag{3.32}$$

Since $\phi(0) = 0$, (3.32) becomes

$$\phi\left(\int_{\frac{a+b}{2}}^b |D_q h(x)| d_q x\right) \geq \int_{\frac{a+b}{2}}^b D_q \phi(|h(x)|) |D_q h(x)| d_q x. \tag{3.33}$$

By (3.31) and (3.33) we obtain

$$\int_a^b D_q\phi(|h(x)|)|D_qh(x)|d_qx \leq \phi\left(\int_a^{\frac{a+b}{2}} |D_qh(x)|d_qx\right) + \phi\left(\int_{\frac{a+b}{2}}^b |D_qh(x)|d_qx\right). \tag{3.34}$$

Now, for $\phi(x) = \frac{x^p}{p}$, $1 \leq p < \infty$ in (3.34) we have

$$\frac{[p]_q}{p}\int_a^b |D_qh(x)||h(x)|^{p-1}d_qx \leq \frac{1}{p}\left(\int_a^{\frac{a+b}{2}} |D_qh(x)|d_qx\right)^p + \frac{1}{p}\left(\int_{\frac{a+b}{2}}^b |D_qh(x)|d_qx\right)^p. \tag{3.35}$$

By Lemma 3.1, we obtain

$$\frac{[p]_q}{p}\int_a^b |D_qh(x)||h(x)|^{p-1}d_qx \leq \frac{(b-a)^{p-1}}{2^{p-1}p}\int_a^{\frac{a+b}{2}} |D_qh(x)|^pd_qx + \frac{(b-a)^{p-1}}{2^{p-1}p}\int_{\frac{a+b}{2}}^b |D_qh(x)|^pd_qx, \tag{3.36}$$

which simplifies to

$$\int_a^b |D_qh(x)||h(x)|^{p-1}d_qx \leq \frac{(b-a)^{p-1}}{2^{p-1}[p]_q}\int_a^b |D_qh(x)|^pd_qx.$$

This completes the proof of the theorem. □

Remark 3.5. The constant $\frac{(b-a)^{p-1}}{2^{p-1}[p]_q}$ is sharper than the constant $\frac{(b-a)^{p-1}}{[p]_q}$.

Remark 3.6. Taking $p = 2$ in (3.24) yields

$$\int_a^b |D_qh(x)||h(x)|d_qx \leq \frac{(b-a)}{2[2]_q}\int_a^b |D_qh(x)|^2d_qx. \tag{3.37}$$

This simplifies to

$$\int_a^b |D_qh(x)||h(x)|d_qx \leq \frac{(1-q)(b-a)}{2(1-q^2)}\int_a^b |D_qh(x)|^2d_qx = \frac{(b-a)}{2(1+q)}\int_a^b |D_qh(x)|^2d_qx, \tag{3.38}$$

as the q-analogue of (1.3).

Remark 3.7. By taking the limit as $q \to 1^-$ in (3.38) yields (1.3).

4 Conclusion

In this paper, interesting results on q-analogues of generalized Opial's inequalities were established and also presented some extensions of the analogues. The basic concepts of q-calculus, convexity properties of functions and the application of the Hölder's integral inequality were employed to obtain these results.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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