Closed Formulas for the Sums of Squares of Generalized Fibonacci Numbers

Yüksel Soykan[1*]

[1]*Department of Mathematics, Faculty of Art and Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Original Research Article

ABSTRACT

In this paper, closed forms of the sum formulas for the squares of generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Our work generalize second order recurrence relations.

1 INTRODUCTION

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are very well-known examples of second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem,

---

Corresponding author: E-mail: yuksel_soykan@hotmail.com;

introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci sequences are a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence.

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \ \ n \geq 2, \ \ \ F_0 = 0, \ F_1 = 1.$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \ \ n \geq 2, \ \ \ L_0 = 2, \ L_1 = 1.$$

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. In 1965, Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \ W_0 = a, \ W_1 = b, \ \ (n \geq 2) \ \ (1.1)$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	Notation: $W_n(a, b; r, s)$	OEIS: [5]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

and

$$\sum_{i=1}^{n} Q_i^2 = \frac{1}{2}(Q_n Q_{n+1} - 4).$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. Our results generalize second order linear recurrence relations.

We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[6,7,8,9,10]	[11,12]	[13,14,15]
Generalized Tribonacci	[16]		
Generalized Tetranacci	[17,18]		

An application of the sum of the squares of the numbers is circulant matrix. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized m-step Fibonacci sequences see for example Soykan [19]. If $m = 2, m = 3$ and $m = 4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively. Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm.

Circulant matrices have been around for a long time and have been extensively used in many scientific areas. In some scientific areas such as image processing, coding theory and signal processing we often encounter circulant matrices. These matrices also have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

Let $n \geq 2$ be an integer and r be any real or complex number. An $n \times n$ matrix C_r is called a r-circulant matrix if it of the form

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n \times n}.$$

and the r-circulant matrix C_r is denoted by $C_r = Circ_r(c_0, c_1, ..., c_{n-1})$. If $r = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = Circ(c_0, c_1, ..., c_{n-1})$. Circulant matrix was first proposed by Davis in [20]. This matrix has many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [21] studied on the norms of r-circulant matrices with Fibonacci and Lucas numbers. Then, later Kızılateş and Tuglu [22] defined a new geometric circulant matrix as follows:

$$C_{r^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n \times n}.$$

and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci number and Lucas numbers. When the parameter satisfies $r = 1$, we get the classical circulant matrix. See also Polatlı [23] for the spectral norms of r-circulant matrices with a type of Catalan triangle numbers.

The Frobenius (or Euclidean) norm and spectral norm of a matrix $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^{1/2}$$

where λ_i's are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A. The maximum column length norm $c_1(.)$ and the maximum row length norm $r_1(.)$ of an matrix of

order $n \times n$ are defined as follows:

$$c_1(A) = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \text{ and } r_1(A) = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

The following inequality holds for any matrix $A = M_{n \times n}(\mathbb{C})$:

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F.$$

Calculations of the above norms $\|A\|_F, \|A\|_2, c_1(A)$ and $r_1(A)$ require the sum of the squares of the numbers a_{ij}. As in our case, the numbers a_{ij} can be chosen as elements of second, third or higher order linear recurrence sequences.

In the following Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$).

Table 3. Papers on the norms.

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[24,22,25,26,27,28,29,21,30,31,32,33]
Pell, Pell-Lucas	[34,35]
Jacobsthal, Jacobsthal-Lucas	[36,37,38,39]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[40,41]
Padovan, Perrin	[42,43,44]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[45]

Also linear summing formulas of the generalized m-step Fibonacci sequences are required for the computation of various norms of circulant matrices circulant matrices with the generalized m-step Fibonacci sequences. We present some works on summing formulas of the numbers in the following Table 4.

Table 4. A few special study of sum formulas.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[46],[47,48]
Generalized Fibonacci	[49,50,51,52,53]
Generalized Tribonacci	[54,55,55,56]
Generalized Tetranacci	[57,58,59]
Generalized Pentanacci	[60,61]
Generalized Hexanacci	[62]

2   SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 1$ we have the following formulas: if $(s+1)(r+s-1)(r-s+1) \neq 0$ then*
(a)

$$\sum_{i=1}^{n} W_i^2 = \frac{(1-s)W_{n+2}^2 + (1-s-r^2-r^2 s)W_{n+1}^2 + 2rs W_{n+1}W_{n+2} + (s-1)W_1^2 + s^2(s-1)W_0^2 - 2rs W_1 W_0}{(s+1)(r+s-1)(r-s+1)}.$$

(b)

$$\sum_{i=1}^{n} W_{i+1}W_i = \frac{r W_{n+2}^2 + rs^2 W_{n+1}^2 + (1-r^2-s^2)W_{n+1}W_{n+2} - r W_1^2 - rs^2 W_0^2 + s(-r^2+s^2-1)W_1 W_0}{(s+1)(r+s-1)(r-s+1)}.$$

Proof. Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$
\begin{aligned}
s^2 W_n^2 &= W_{n+2}^2 + r^2 W_{n+1}^2 - 2r W_{n+2}W_{n+1} \\
s^2 W_{n-1}^2 &= W_{n+1}^2 + r^2 W_n^2 - 2r W_{n+1}W_n \\
s^2 W_{n-2}^2 &= W_n^2 + r^2 W_{n-1}^2 - 2r W_n W_{n-1} \\
s^2 W_{n-3}^2 &= W_{n-1}^2 + r^2 W_{n-2}^2 - 2r W_{n-1}W_{n-2} \\
s^2 W_{n-4}^2 &= W_{n-2}^2 + r^2 W_{n-3}^2 - 2r W_{n-2}W_{n-3}
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
s^2 W_2^2 &= W_4^2 + r^2 W_3^2 - 2r W_4 W_3 \\
s^2 W_1^2 &= W_3^2 + r^2 W_2^2 - 2r W_3 W_2.
\end{aligned}
$$

If we add the above equations by side by, we get

$$s^2 \sum_{i=1}^{n} W_i^2 = \sum_{i=3}^{n+2} W_i^2 + r^2 \sum_{i=2}^{n+1} W_i^2 - 2r \sum_{i=2}^{n+1} W_{i+1}W_i. \tag{2.1}$$

Note that

$$
\begin{aligned}
\sum_{i=3}^{n+2} W_i^2 &= -W_1^2 - W_2^2 + W_{n+1}^2 + W_{n+2}^2 + \sum_{i=1}^{n} W_i^2, \\
\sum_{i=2}^{n+1} W_i^2 &= -W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2, \\
\sum_{i=2}^{n+1} W_{i+1}W_i &= -W_2 W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i.
\end{aligned}
$$

We put them into the (2.1) we obtain

$$
\begin{aligned}
s^2 \sum_{i=1}^{n} W_i^2 = {}& \left(-W_1^2 - W_2^2 + W_{n+1}^2 + W_{n+2}^2 + \sum_{i=1}^{n} W_i^2\right) + r^2\left(-W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2\right) \\
& -2r\left(-W_2 W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i\right).
\end{aligned} \tag{2.2}
$$

Next we calculate $\sum_{i=1}^{n} W_{i+1}W_i$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$
\begin{aligned}
sW_{n+1}W_n &= W_{n+2}W_{n+1} - rW_{n+1}^2 \\
sW_nW_{n-1} &= W_{n+1}W_n - rW_n^2 \\
sW_{n-1}W_{n-2} &= W_nW_{n-1} - rW_{n-1}^2 \\
&\vdots \\
sW_3W_2 &= W_4W_3 - rW_3^2 \\
sW_2W_1 &= W_3W_2 - rW_2^2.
\end{aligned}
$$

If we add the above equations by side by, we get

$$s\sum_{i=1}^{n} W_{i+1}W_i = \sum_{i=2}^{n+1} W_{i+1}W_i - r\sum_{i=2}^{n+1} W_i^2. \tag{2.3}$$

Note that

$$
\begin{aligned}
\sum_{i=2}^{n+1} W_{i+1}W_i &= -W_2W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i, \\
\sum_{i=2}^{n+1} W_i^2 &= -W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2.
\end{aligned}
$$

If we put them into the (2.3) then it follows that

$$s\sum_{i=1}^{n} W_{i+1}W_i = \left(-W_2W_1 + W_{n+2}W_{n+1} + \sum_{i=1}^{n} W_{i+1}W_i\right) - r\left(-W_1^2 + W_{n+1}^2 + \sum_{i=1}^{n} W_i^2\right). \tag{2.4}$$

Then, using

$$W_2 = (rW_1 + sW_0)$$

and solving the system (2.2)-(2.4), the required results of (a) and (b) follow.
Taking $r = s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.1. *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{i=1}^{n} W_i^2 = \frac{1}{2}(-2W_{n+1}^2 + 2W_{n+2}W_{n+1} - 2W_1W_0)$.

(b) $\sum_{i=1}^{n} W_{i+1}W_i = \frac{1}{2}(W_{n+2}^2 + W_{n+1}^2 - W_{n+1}W_{n+2} - W_1^2 - W_0^2 - W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.2. *For $n \geq 1$, Fibonacci numbers have the following properties:*

(a) $\sum_{i=1}^{n} F_i^2 = \frac{1}{2}(-2F_{n+1}^2 + 2F_{n+2}F_{n+1})$.

(b) $\sum_{i=1}^{n} F_{i+1}F_i = \frac{1}{2}(F_{n+2}^2 + F_{n+1}^2 - F_{n+1}F_{n+2} - 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.3. *For $n \geq 1$, Lucas numbers have the following properties:*

(a) $\sum_{i=1}^{n} L_i^2 = \frac{1}{2}(-2L_{n+1}^2 + 2L_{n+2}L_{n+1} - 4)$.

(b) $\sum_{i=1}^{n} L_{i+1}L_i = \frac{1}{2}(L_{n+2}^2 + L_{n+1}^2 - L_{n+1}L_{n+2} - 7)$.

Taking $r = 2, s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.2. *If* $r = 2, s = 1$ *then for* $n \geq 0$ *we have the following formulas:*

(a) $\sum_{i=1}^{n} W_i^2 = \frac{1}{2}(-2W_{n+1}^2 + W_{n+2}W_{n+1} - W_1W_0)$.

(b) $\sum_{i=1}^{n} W_{i+1}W_i = \frac{1}{4}(W_{n+2}^2 + W_{n+1}^2 - 2W_{n+2}W_{n+1} - W_1^2 - W_0^2 - 2W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.4. *For* $n \geq 1$, *Pell numbers have the following properties:*

(a) $\sum_{i=1}^{n} P_i^2 = \frac{1}{2}(-2P_{n+1}^2 + P_{n+2}P_{n+1}) = \frac{1}{2}P_nP_{n+1}$.

(b) $\sum_{i=1}^{n} P_{i+1}P_i = \frac{1}{4}(P_{n+2}^2 + P_{n+1}^2 - 2P_{n+2}P_{n+1} - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.5. *For* $n \geq 1$, *Pell-Lucas numbers have the following properties:*

(a) $\sum_{i=1}^{n} Q_i^2 = \frac{1}{2}(-2Q_{n+1}^2 + Q_{n+2}Q_{n+1} - 4) = \frac{1}{2}(Q_nQ_{n+1} - 4)$.

(b) $\sum_{i=1}^{n} Q_{i+1}Q_i = \frac{1}{4}(Q_{n+2}^2 + Q_{n+1}^2 - 2Q_{n+2}Q_{n+1} - 16)$.

If $r = 1, s = 2$ then $(s + 1)(r + s - 1)(r - s + 1) = 0$ so we can't use Theorem 2.1 directly. Therefore we need another method to find $\sum_{i=1}^{n} W_i^2$ and $\sum_{i=1}^{n} W_{i+1}W_i$ which is given in the following theorem.

Theorem 2.6. *If* $r = 1, s = 2$ *then for* $n \geq 1$ *we have the following formulas:*

(a) $\sum_{i=1}^{n} W_i^2 = \frac{1}{9}(W_{n+2}^2 - W_{n+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 n)$.

(b) $\sum_{i=1}^{n} W_{i+1}W_i = \frac{1}{36}(5W_{n+2}^2 + 4W_{n+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1W_0) - 4(W_1 - 2W_0)^2 n)$.

Proof.

(a) The proof will be by induction on n. Before the proof, we recall some information on generalized Jacobsthal numbers. A generalized Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} + 2W_{n-2}; \quad W_0 = a, \ W_1 = b, \quad (n \geq 2) \tag{2.5}$$

with the initial values W_0, W_1 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (2.5) holds for all integer n. The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 5.

Table 5. A few generalized Jacobsthal numbers

n	W_n	W_{-n}
0	W_0	
1	W_1	$-\frac{1}{2}W_0 + \frac{1}{2}W_1$
2	$2W_0 + W_1$	$\frac{3}{4}W_0 - \frac{1}{4}W_1$
3	$2W_0 + 3W_1$	$-\frac{5}{8}W_0 + \frac{3}{8}W_1$
4	$6W_0 + 5W_1$	$\frac{11}{16}W_0 - \frac{5}{16}W_1$
5	$10W_0 + 11W_1$	$-\frac{21}{32}W_0 + \frac{11}{32}W_1$
6	$22W_0 + 21W_1$	$\frac{43}{64}W_0 - \frac{21}{64}W_1$

Binet formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as

$$t^2 - t - 2 = 0.$$

The roots of characteristic equation are

$$\alpha = 2, \;\; \beta = -1$$

and the roots satisfy the following

$$\alpha + \beta = 1, \; \alpha\beta = -2, \; \alpha - \beta = 3.$$

Using these roots and the recurrence relation, Binet formula can be given as

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} = \frac{A \times 2^n - B(-1)^n}{3} \tag{2.6}$$

where $A = W_1 - W_0\beta = W_1 + W_0$ and $B = W_1 - W_0\alpha = W_1 - 2W_0$.

We now prove (a) by induction on n. If $n = 1$ we see that the sum formula reduces to the relation

$$W_1^2 = \frac{1}{9}(W_3^2 - W_2^2 + W_1^2 - 8W_1W_0). \tag{2.7}$$

Since

$$\begin{aligned} W_2 &= 2W_0 + W_1, \\ W_3 &= 2W_0 + 3W_1, \end{aligned}$$

(2.7) is true. Assume that the relation in (a) is true for $n = m$, i.e.,

$$\sum_{i=1}^{m} W_i^2 = \frac{1}{9}(W_{m+2}^2 - W_{m+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 m).$$

Then we get

$$\begin{aligned} \sum_{i=1}^{m+1} W_i^2 &= W_{m+1}^2 + \sum_{i=1}^{m} W_i^2 \\ &= W_{m+1}^2 + \frac{1}{9}(W_{m+2}^2 - W_{m+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 m) \\ &= \frac{1}{9}(W_{m+2}^2 + 8W_{m+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 m) \\ &= \frac{1}{9}(W_{m+2}^2 + 8W_{m+1}^2 - (2W_0 - W_1)^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 (m+1)) \\ &= \frac{1}{9}(W_{m+3}^2 - W_{m+2}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 (m+1)) \\ &= \frac{1}{9}(W_{(m+1)+2}^2 - W_{(m+1)+1}^2 - 4(W_0 + W_1)W_0 + (2W_0 - W_1)^2 (m+1)) \end{aligned}$$

where

$$W_{m+2}^2 + 8W_{m+1}^2 - (2W_0 - W_1)^2 = W_{m+3}^2 - W_{m+2}^2. \tag{2.8}$$

(2.8) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n = m + 1$.

(b) We now prove (b) by induction on n. If $n = 1$ we see that the sum formula reduces to the relation

$$W_2 W_1 = \frac{1}{36}(5W_3^2 + 4W_2^2 - 13W_1^2 - 36W_0^2 - 4W_1 W_0). \tag{2.9}$$

Since

$$W_2 = 2W_0 + W_1,$$
$$W_3 = 2W_0 + 3W_1,$$

(2.9) is true. Assume that the relation in (b) is true for $n = m$, i.e.,

$$\sum_{i=1}^{m} W_{i+1} W_i = \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1 W_0) - 4(W_1 - 2W_0)^2 m).$$

Then we get

$$\begin{aligned}
\sum_{i=1}^{m+1} W_{i+1} W_i &= W_{m+2} W_{m+1} + \sum_{i=1}^{m} W_{i+1} W_i \\
&= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2} W_{m+1} + (-9W_1^2 - 20W_0^2 - 20W_1 W_0) - 4(W_1 - 2W_0)^2 m) \\
&= \frac{1}{36}(5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2} W_{m+1} + 4(W_1 - 2W_0)^2 + (-9W_1^2 - 20W_0^2 - 20W_1 W_0) \\
&\quad -4(W_1 - 2W_0)^2 (m + 1)) \\
&= \frac{1}{36}(5W_{m+3}^2 + 4W_{m+2}^2 + (-9W_1^2 - 20W_0^2 - 20W_1 W_0) - 4(W_1 - 2W_0)^2 (m + 1)) \\
&= \frac{1}{36}(5W_{(m+1)+2}^2 + 4W_{(m+1)+1}^2 + (-9W_1^2 - 20W_0^2 - 20W_1 W_0) - 4(W_1 - 2W_0)^2 (m + 1))
\end{aligned}$$

where

$$5W_{m+2}^2 + 4W_{m+1}^2 + 36W_{m+2} W_{m+1} + 4(W_1 - 2W_0)^2 = 5W_{m+3}^2 + 4W_{m+2}^2. \tag{2.10}$$

(2.10) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n = m + 1$.

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.7. *For $n \geq 1$, Jacobsthal numbers have the following property:*

(a) $\sum_{i=1}^{n} J_i^2 = \frac{1}{9}(J_{n+2}^2 - J_{n+1}^2 + n)$.
(b) $\sum_{i=1}^{n} J_{i+1} J_i = \frac{1}{36}(5J_{n+2}^2 + 4J_{n+1}^2 - 9 - 4n)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.8. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{i=1}^{n} j_i^2 = \frac{1}{9}(j_{n+2}^2 - j_{n+1}^2 - 24 + 9n)$.
(b) $\sum_{i=1}^{n} j_{i+1} j_i = \frac{1}{36}(5j_{n+2}^2 + 4j_{n+1}^2 - 129 - 36n)$.

3 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas: If $(s+1)(r+s-1)(r-s+1) \neq 0$ then*

(a)

$$\sum_{i=1}^{n} W_{-i}^2 = \frac{\begin{array}{c}(s-1)W_{-n+1}^2 + (r^2+r^2s+s-1)W_{-n}^2 - 2rsW_{-n+1}W_{-n} + 2rsW_1W_0 + (1-s)W_1^2 \\ +(1-s-r^2-r^2s)W_0^2\end{array}}{(s+1)(r+s-1)(r-s+1)}.$$

(b)

$$\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{-rW_{-n+1}^2 - rs^2W_{-n}^2 + (r^2+s^2-1)W_{-n+1}W_{-n} + (1-r^2-s^2)W_1W_0 + rW_1^2 + rs^2W_0^2}{(s+1)(r+s-1)(r-s+1)}.$$

Proof. Using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$
\begin{array}{rcl}
s^2W_{-n}^2 & = & W_{-n+2}^2 + r^2W_{-n+1}^2 - 2rW_{-n+2}W_{-n+1} \\
s^2W_{-n+1}^2 & = & W_{-n+3}^2 + r^2W_{-n+2}^2 - 2rW_{-n+3}W_{-n+2} \\
s^2W_{-n+2}^2 & = & W_{-n+4}^2 + r^2W_{-n+3}^2 - 2rW_{-n+4}W_{-n+3} \\
s^2W_{-n+3}^2 & = & W_{-n+5}^2 + r^2W_{-n+4}^2 - 2rW_{-n+5}W_{-n+4} \\
& \vdots & \\
s^2W_{-3}^2 & = & W_{-1}^2 + r^2W_{-2}^2 - 2rW_{-1}W_{-2} \\
s^2W_{-2}^2 & = & W_0^2 + r^2W_{-1}^2 - 2rW_0W_{-1} \\
s^2W_{-1}^2 & = & W_1^2 + r^2W_0^2 - 2rW_1W_0.
\end{array}
$$

If we add the above equations by side by, we get

$$
\begin{aligned}
s^2 \sum_{i=1}^{n} W_{-i}^2 = {} & (W_1^2 + W_0^2 - W_{-n+1}^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2) + r^2(W_0^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2) \quad (3.1) \\
& -2r(W_1W_0 - W_{-n+1}W_{-n} + \sum_{i=1}^{n} W_{-i+1}W_{-i})
\end{aligned}
$$

Next we calculate $\sum_{i=1}^{n} W_{-i+1}W_{-i}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$
\begin{aligned}
sW_{-n+1}W_{-n} &= W_{-n+2}W_{-n+1} - rW_{-n+1}^2 \\
sW_{-n+2}W_{-n+1} &= W_{-n+3}W_{-n+2} - rW_{-n+2}^2 \\
sW_{-n+3}W_{-n+2} &= W_{-n+4}W_{-n+3} - rW_{-n+3}^2 \\
sW_{-n+4}W_{-n+3} &= W_{-n+5}W_{-n+4} - rW_{-n+4}^2 \\
&\vdots \\
sW_{-2}W_{-3} &= W_{-1}W_{-2} - rW_{-2}^2 \\
sW_{-1}W_{-2} &= W_0 W_{-1} - rW_{-1}^2 \\
sW_0 W_{-1} &= W_1 W_0 - rW_0^2.
\end{aligned}
$$

If we add the above equations by side by, we get

$$
s\sum_{i=1}^{n} W_{-i+1}W_{-i} = (W_1 W_0 - W_{-n+1}W_{-n} + \sum_{i=1}^{n} W_{-i+1}W_{-i}) - r(W_0^2 - W_{-n}^2 + \sum_{i=1}^{n} W_{-i}^2). \quad (3.2)
$$

Then, solving the system (3.1)-(3.2), the required results of (a) and (b) follow.

Taking $r = s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.1. *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{i=1}^{n} W_{-i}^2 = \frac{1}{2}(2W_{-n}^2 - 2W_{-n+1}W_{-n} + 2W_1 W_0 - 2W_0^2).$
(b) $\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{2}(-W_{-n+1}^2 - W_{-n}^2 + W_{-n+1}W_{-n} - W_1 W_0 + W_1^2 + W_0^2).$

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.2. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{i=1}^{n} F_{-i}^2 = \frac{1}{2}(2F_{-n}^2 - 2F_{-n+1}F_{-n}).$
(b) $\sum_{i=1}^{n} F_{-i+1}F_{-i} = \frac{1}{2}(-F_{-n+1}^2 - F_{-n}^2 + F_{-n+1}F_{-n} + 1).$

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.3. *For $n \geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{i=1}^{n} L_{-i}^2 = \frac{1}{2}(2L_{-n}^2 - 2L_{-n+1}L_{-n} - 4).$
(b) $\sum_{i=1}^{n} L_{-i+1}L_{-i} = \frac{1}{2}(-L_{-n+1}^2 - L_{-n}^2 + L_{-n+1}L_{-n} + 3).$

Taking $r = 2, s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.2. *If $r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{i=1}^{n} W_{-i}^2 = \frac{1}{2}(2W_{-n}^2 - W_{-n+1}W_{-n} - 2W_0^2 + W_1 W_0).$
(b) $\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{4}(-W_{-n+1}^2 - W_{-n}^2 + 2W_{-n+1}W_{-n} + W_1^2 + W_0^2 - 2W_1 W_0).$

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 3.4. *For $n \geq 1$, Pell numbers have the following properties.*

(a) $\sum_{i=1}^{n} P_{-i}^2 = \frac{1}{2}(2P_{-n}^2 - P_{-n+1}P_{-n})$.

(b) $\sum_{i=1}^{n} P_{-i+1}P_{-i} = \frac{1}{4}(-P_{-n+1}^2 - P_{-n}^2 + 2P_{-n+1}P_{-n} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.5. *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a) $\sum_{i=1}^{n} Q_{-i}^2 = \frac{1}{2}(2Q_{-n}^2 - Q_{-n+1}Q_{-n} - 4)$.

(b) $\sum_{i=1}^{n} Q_{-i+1}Q_{-i} = \frac{1}{4}(-Q_{-n+1}^2 - Q_{-n}^2 + 2Q_{-n+1}Q_{-n})$.

If $r = 1, s = 2$ then $(s+1)(r+s-1)(r-s+1) = 0$ so we can't use Theorem 3.1 directly. Therefore we need another method to find $\sum_{i=1}^{n} W_{-i}^2$ and $\sum_{i=1}^{n} W_{-i+1}W_{-i}$ which is given in the following theorem.

Theorem 3.6. *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{i=1}^{n} W_{-i}^2 = \frac{1}{9}(-W_{-n+1}^2 + W_{-n}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 n)$.

(b) $\sum_{i=1}^{n} W_{-i+1}W_{-i} = \frac{1}{27}(-2W_{-n+1}^2 + 4W_{-n}^2 - 7W_{-n+1}W_{-n} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 n)$.

Proof. (a) and (b) can be proved by mathematical induction.

(a) The proof will be by induction on n. If $n = 1$ we see that the sum formula reduces to the relation

$$W_{-1}^2 = \frac{1}{9}(2W_0^2 - 4W_0W_1 + 2W_1^2 + W_{-1}^2). \tag{3.3}$$

Since

$$W_{-1} = (-\frac{1}{2}W_0 + \frac{1}{2}W_1)$$

(3.3) is true. Assume that the relation in (a) is true for $n = m$, i.e.

$$\sum_{i=1}^{m} W_{-i}^2 = \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 m).$$

Then we get

$$
\begin{aligned}
\sum_{i=1}^{m+1} W_{-i}^2 &= W_{-(m+1)}^2 + \sum_{i=1}^{m} W_{-i}^2 \\
&= W_{-m-1}^2 + \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 m) \\
&= \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + 9W_{-m-1}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 m) \\
&= \frac{1}{9}(-W_{-m+1}^2 + W_{-m}^2 + 9W_{-m-1}^2 - (W_1 - 2W_0)^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{9}(-W_{-m}^2 + W_{-m-1}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 (m+1)) \\
&= \frac{1}{9}(-W_{-(m+1)+1}^2 + W_{-(m+1)}^2 + (W_1^2 - W_0^2) + (W_1 - 2W_0)^2 (m+1))
\end{aligned}
$$

where

$$-W_{-m+1}^2 + W_{-m}^2 + 9W_{-m-1}^2 - (W_1 - 2W_0)^2 = -W_{-m}^2 + W_{-m-1}^2. \tag{3.4}$$

(3.4) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n = m + 1$.

(b) We now prove (b) by induction on n. If $n = 1$ we see that the sum formula reduces to the relation

$$W_0 W_{-1} = \frac{1}{27}(-W_1^2 - 18W_0^2 + 4W_{-1}^2 + 19W_0 W_1 - 7W_0 W_{-1}). \tag{3.5}$$

Since

$$W_{-1} = (-\frac{1}{2}W_0 + \frac{1}{2}W_1),$$

(3.5) is true. Assume that the relation in (b) is true for $n = m$ i.e.,

$$\sum_{i=1}^{m} W_{-i+1} W_{-i} = \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 m).$$

Then we get

$$
\begin{aligned}
\sum_{i=1}^{m+1} W_{-i+1}W_{-i} &= W_{-(m+1)+1}W_{-(m+1)} + \sum_{i=1}^{m} W_{-i+1}W_{-i} \\
&= W_{-m}W_{-m-1} + \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} \\
&\quad + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 m) \\
&= \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} \\
&\quad + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 m) \\
&= \frac{1}{27}(-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} + 3(W_1 - 2W_0)^2 \\
&\quad + (W_1 + 4W_0)(2W_1 - W_0) - 3(W_1 - 2W_0)^2 (m + 1)) \\
&= \frac{1}{27}(-2W_{-m}^2 + 4W_{-m-1}^2 - 7W_{-m}W_{-m-1} + (W_1 + 4W_0)(2W_1 - W_0) \\
&\quad - 3(W_1 - 2W_0)^2 (m + 1)) \\
&= \frac{1}{27}(-2W_{-(m+1)+1}^2 + 4W_{-(m+1)}^2 - 7W_{-(m+1)+1}W_{-(m+1)} + (W_1 + 4W_0)(2W_1 - W_0) \\
&\quad - 3(W_1 - 2W_0)^2 (m + 1))
\end{aligned}
$$

where

$$-2W_{-m+1}^2 + 4W_{-m}^2 - 7W_{-m+1}W_{-m} + 27W_{-m}W_{-m-1} + 3(W_1 - 2W_0)^2 = -2W_{-m}^2 + 4W_{-m-1}^2 - 7W_{-m}W_{-m-1}. \tag{3.6}$$

(3.6) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n = m + 1$.

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.7. *For $n \geq 1$, Jacobsthal numbers have the following property:*

(a) $\sum_{i=1}^{n} J_{-i}^2 = \frac{1}{9}(-J_{-n+1}^2 + J_{-n}^2 + 1 + n)$.

(b) $\sum_{i=1}^{n} J_{-i+1}J_{-i} = \frac{1}{27}(-2J_{-n+1}^2 + 4J_{-n}^2 - 7J_{-n+1}J_{-n} + 2 - 3n)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.8. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{i=1}^{n} j_{-i}^2 = \frac{1}{9}(-j_{-n+1}^2 + j_{-n}^2 - 3 + 9n)$.

(b) $\sum_{i=1}^{n} j_{-i+1}j_{-i} = \frac{1}{27}(-2j_{-n+1}^2 + 4j_{-n}^2 - 7j_{-n+1}j_{-n} - 27n)$.

4   CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. Our future work will be investigation of the closed forms of the sum formulas for the squares of generalized Tribonacci numbers.

COMPETING INTERESTS

Authors has declared that no competing interests exist.

REFERENCES

[1]   Horadam AF. Basic properties of a certain generalized sequence of numbers. Fibonacci Quarterly. 1965;3(3):161-176.

[2]   Horadam AF. A generalized fibonacci sequence. American Mathematical Monthly. 1961;68:455-459.

[3]   Horadam AF. Special Properties of The Sequence $w_n(a, b; p, q)$. Fibonacci Quarterly. 1967;5(5):424-434.

[4]   Horadam AF. Generating functions for powers of a certain generalized sequence of numbers. Duke Math. J. 1965;32:437-446.

[5]   Sloane NJA. The on-line encyclopedia of integer sequences.
Available: http://oeis.org/

[6]   Čerin Z. Formulae for sums of Jacobsthal–Lucas numbers. Int. Math. Forum. 2007;2(40):1969–1984.

[7]   Čerin Z. Sums of squares and products of jacobsthal numbers. Journal of Integer Sequences. 2007;10. Article 07.2.5, 2007

[8]   Gnanam A, Anitha B. Sums of squares Jacobsthal numbers. IOSR Journal of Mathematics 2015;11(6):62-64.

[9]   Kiliç E, Taşçi, D. The linear algebra of the pell matrix. Boletín de la Sociedad Matemática Mexicana. 2005;3(11).

[10]   Kılıc E. Sums of the squares of terms of sequence $\{u_n\}$. Proc. Indian Acad. Sci. (Math. Sci.) 2008;118(1):27–41.

[11]   Frontczak R. Sums of cubes over odd-index fibonacci numbers. Integers. 2018;18.

[12]   Wamiliana, Suharsono, Kristanto PE. Counting the sum of cubes for Lucas and Gibonacci Numbers, Science and Technology Indonesia. 2019;4(2):31-35.

[13]   Chen L, Wang X. The power sums involving fibonacci polynomials and their applications. Symmetry. 2019;11.
DOI: 10.3390/sym11050635

[14]   Frontczak, R. Sums of powers of Fibonacci and Lucas numbers: A new bottom-up Approach. Notes on Number Theory and Discrete Mathematics. 2018;24(2):94–103.

[15]   Prodinger H. Sums of powers of fibonacci polynomials. Proc. Indian Acad. Sci. (Math. Sci.). 2009;119(5):567-570.

[16]   Raza Z, Riaz M, Ali MA. Some inequalities on the norms of special matrices with generalized tribonacci and generalized pell-padovan sequences; 2015. arXiv
Availabe: http://arxiv.org/abs/1407.1369v2

[17]   Prodinger H, Selkirk SJ. Sums of squares of tetranacci numbers: A generating function approach; 2019.
Available: http://arxiv.org/abs/1906.08336v1

[18] Schumacher R. How to sum the squares of the Tetranacci numbers and the Fibonacci m-step numbers. Fibonacci Quarterly. 2019;57:168-175.

[19] Soykan Y. Simson identity of generalized m-step fibonacci numbers. Int. J. Adv. Appl. Math. and Mech. 2019;7(2):45-56.

[20] Davis PJ. Circulant matrices. JohnWiley&Sons, New York; 1979.

[21] Shen S, Cen J. On the bounds for the norms of r-circulant matrices with the fibonacci and lucas numbers. Applied Mathematics and Computation. 2010;216;2891-2897.

[22] KızılateşC, Tuglu N. On the bounds for the spectral norms of geometric circulant matrices. Journal of Inequalities and Applications. 2016(1). Article ID 312, 2016. DOI: 10.1186/s13660-016-1255-1

[23] PolatlıE. On the bounds for the spectral norms of r-circulant matrices with a type of catalan triangle numbers. Journal of Science and Arts. 2019;3(48):575-578.

[24] He C, Ma J. Zhang K, Wang Z. The upper bound estimation on the spectral norm of r-circulant matrices with the fibonacci and lucas numbers. J. Inequal. Appl. 2015;2015:72.

[25] Merikoski JK, Haukkanen P, Mattila M, Tossavainen T. On the spectral and frobenius norm of a generalized fibonacci r-circulant matrix. Special Matrices. 2018;6:23-36.

[26] Raza Z, Ali MA. On the norms of some special matrices with generalized fibonacci sequence. J. Appl. Math. & Informatics. 2015;33(5-6):593-605.

[27] Shen S. On the norms of circulant matrices with the (k,h)-fibonacci and (k,h)-lucas numbers. Int. J. Contemp. Math. Sciences. 2011;6(18):887-894.

[28] Shen S. The spectral norms of circulant matrices involving (k,h)-fibonacci and (k,h)-lucas numbers. Int. J. Contemp. Math. Sciences. 2014;9(14):661-665.

[29] Shen S, Cen J. On the spectral norms of r-circulant matrices with the k-fibonacci and k-lucas numbers. Int. J. Contemp. Math. Sciences. 2010;5(12):569-578.

[30] Shi B. The spectral norms of geometric circulant matrices with the generalized k-horadam numbers. Journal of Inequalities and Applications. 2018;2018:14. Available: https://doi.org/10.1186/s13660-017-1608-4

[31] Solak S. On the norms of circulant matrices with the fibonacci and lucas numbers. Applied Mathematics and Computation. 2005;160:125-132.

[32] Solak S. Erratum to "On the norms of circulant matrices with the fibonacci and lucas numbers" [Appl. Math. Comput. 2005;160: 125-132], Applied Mathematics and Computation. 2007;190:1855-856.

[33] Tuglu N, Kızılateş C. On the norms of circulant and r-circulant matrices with the hyperharmonic fibonacci numbers. Journal of Inequalities and Applications; 2015. Article ID 253, 2015 Available: http://dx.doi.org/10.1186/s13660-015-0778-1

[34] Alptekin E. G.Pell, Pell-Lucas ve Modified Pell sayıları ile tanımlı circulant ve semicirculant matrisler (Doctoral dissertation, Selçuk Üniversitesi Fen Bilimleri Enstitüsü); 2005.

[35] Turkmen R, GökbaşH. On the spectral norm of r-circulant matrices with the pell and pell-lucas numbers. J. Inequal. Appl. 2016;2016:65.

[36] Radicic B. On k-circulant matrices involving the jacobsthal numbers. Revista De La Union Matematica Argentina. 2019;60(2):431-442.

[37] Uygun Ş. Some bounds for the norms of circulant matrices with the k-jacobsthal and k-jacobsthal lucas numbers. Journal of Mathematics Research. 2016;8(6):133-138.

[38] Uygun Ş, YaşamalıS. On the bounds for the norms of circulant matrices with the jacobsthal and jacobsthal–lucas numbers. Notes on Number Theory and Discrete Mathematics. 2017;23(1):91-98.

[39] Uygun Ş, YaşamalıS. On the bounds for the norms of r-circulant matrices with the jacobsthal and jacobsthal–lucas numbers. International Journal of Pure and Applied Mathematics. 2017;112(1):93-102.

[40] Kızılateş C, Tuglu N. On the norms of geometric and symmetric geometric circulant matrices with the tribonacci number. Gazi University Journal of Science. 2018;31(2):555-567.

[41] Raza Z, Ali MA. On the norms of circulant, r-circulant. Semi-Circulant and Hankel Matrices with Tribonacci Sequence; 2014. Arxiv
Available: http://arxiv.org/abs/1407.1369v1

[42] Coskun A, Taskara N. On the some properties of circulant matrices with third order linear recurrent sequences. Mathematical Sciences and Applications E-Notes. 2018;6(1):12-18.

[43] Pacheenburawana A, Sintunavarat W. On the spectral norms of r-circulant matrices with the padovan and perrin sequences. Journal of Mathematical Analysis. 2018;9(3):110-122.

[44] Sintunavarat W. The upper bound estimation for the spectral norm of r-circulant and symmetric r-circulant matrices with the padovan sequence. J. Nonlinear Sci. Appl. 2016;9:92-101.

[45] Özkoç A, Ardıyok E. circulant and negacyclic matrices via tetranacci numbers. Honam Mathematical J. 2016;38(4):725-738.

[46] Gökbaş H, Köse H. Some sum formulas for products of pell and pell-lucas numbers. Int. J. Adv. Appl. Math. and Mech. 2017;4(4):1-4.

[47] Koshy T. Fibonacci and lucas numbers with applications. A Wiley-Interscience Publication. New York; 2001.

[48] Koshy T. Pell and pell-lucas numbers with applications. Springer, New York; 2014.

[49] Hansen RT. General identities for linear fibonacci and lucas summations. Fibonacci Quarterly. 1978;16(2):121-28.

[50] Soykan Y. On summing formulas for generalized fibonacci and gaussian generalized fibonacci numbers. Advances in Research. 2019;20(2):1-15.

[51] Soykan Y. Corrigendum: On summing formulas for generalized fibonacci and gaussian generalized fibonacci numbers; 2020.
Available: https://www.researchgate.net/-publication/-337063487_Corrigendum_On-_Summing-_Formulas_For_Generalized-_Fibonacci_and_Gaussian_Generalized-_Fibonacci_Numbers

[52] SoykanY. On summing formulas for horadam numbers. Asian Journal of Advanced Research and Reports. 2020;8(1):45-61.
DOI: 10.9734/AJARR/2020/v8i130192

[53] Soykan Y. Generalized fibonacci numbers: Sum formulas. Journal of Advances in Mathematics and Computer Science. 2020;35(1):89-104.
DOI:10.9734/JAMCS/2020/v35i130241

[54] Frontczak R. Sums of tribonacci and tribonacci-lucas numbers. International Journal of Mathematical Analysis. 2018;12(1):19-24.

[55] Parpar T. k'ncı Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi; 2011. bibitemsoykan2019sumformulatribo Soykan Y. Summing formulas for generalized tribonacci numbers; 2019. arXiv:1910.03490v1 [math.GM].

[56] Soykan Y. Summing formulas for generalized tribonacci numbers. Universal Journal of Mathematics and Applications. Accepted

[57] Soykan Y. Matrix sequences of tribonacci and tribonacci-lucas numbers; 2018. arXiv:1809.07809v1 [math.NT].

[58] Soykan. Summation formulas for generalized tetranacci numbers. Asian Journal of Advanced Research and Reports. 2019;7(2):1-12.
DOI.org/10.9734/ajarr/2019/v7i230170

[59] Waddill ME. The Tetranacci Sequence and Generalizations. Fibonacci Quarterly. 1992;9-20.

[60] Soykan Y. Linear summing formulas of generalized pentanacci and gaussian generalized pentanacci numbers. Journal of Advanced in Mathematics and Computer Science. 2019;33(3):1-14.

[61] Soykan Y. Sum formulas for generalized fifth-order linear recurrence sequences. Journal of Advances in Mathematics and Computer Science. 2019;34(5):1-14. Article no.JAMCS.53303 ISSN: 2456-9968

DOI: 10.9734/JAMCS/2019/v34i530224

[62] Soykan Y. On summing formulas of generalized hexanacci and gaussian generalized hexanacci numbers. Asian Research Journal of Mathematics. 2019;14(4):1-14. Article no.ARJOM.50727

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sdiarticle4.com/review-history/55441