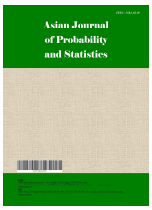

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Properties and Applications of a Transmuted Power Gompertz Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. Authors IBE and AFC designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors BSY and AHB managed the analyses of the study. Author AIA managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This article introduces and studies a new probability distribution called "Transmuted Power Gompertz distribution". It looks at the properties of the transmuted power Gompertz distribution. The article also estimates the four parameters of the new model using the method of maximum likelihood estimation. The article further evaluates the goodness-of-fit of the proposed distribution compared to other distributions by means of applications of the model to two real life datasets and the result show that the proposed distribution is more flexible than the fitted existing distributions.

Keywords: Transmuted power Gompertz distribution; statistical properties; estimation of parameters; application and performance.

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1 Introduction

The Gompertz distribution is both skewed to the right and to the left. It is a generalization of the exponential distribution and is commonly used in many applied problems, particularly in lifetime data analysis [1]. The Gompertz distribution has been applied in the analysis of survival, in some sciences such as gerontology [2], computer [3], biology [4] and marketing science [5]. The hazard rate function of the Gompertz distribution is an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers [6].

New families of distributions are produced day by day and are useful for adding parameters to all forms of probability distributions which makes the resulting distribution more flexible for modeling heavily skewed dataset. Some of these families of distributions include the beta generated family (Beta-G) [7], Transmuted family of distributions [8], Gamma-G (type 1) [9], the Kumaraswamy-G family [10], McDonald-G family [11], Gamma-G (type 2) family [12], Gamma-G (type 3) family [13], Log-gamma-G family [14], Exponentiated T-X family [15], Exponentiated-G (EG) family [16], Weibull-X family [17], Weibull-G family [18], Logistic-G family [19], Gamma-X family [20], a Lomax-G family [21], a new generalized Weibull-G family [22], a Beta Marshall-Olkin family of distributions [23], Logistic-X family [24], a new Weibull-G family [25], a Lindley-G family [26], a Gompertz-G family [27] and Odd Lindley-G family [28] and so on.

As a result of the introduction of these families of probability distribution and the desire to improve the flexibility of classical distributions especially the Gompertz distribution, a number of authors have introduced different extensions of the Gompertz distribution and some of the recent and known studies include the generalized Gompertz distribution [29] which was based on an idea of [30], the Beta Gompertz distribution [31], the odd generalized Exponential-Gompertz distribution [32], the Transmuted Gompertz distribution [33], the Lomax-Gompertz distribution [34] and Power Gompertz distribution [35].

It has been discovered that using the quadratic rank transmutation map by [8] brings about more skewed and flexible probability distributions with an additional shape parameter usually known as the transmuted parameter and this process has led to many flexible compound distributions such as the Transmuted Gompertz distribution [33], the transmuted exponential Lomax distribution by [36], the transmuted odd generalized exponential exponential distribution [37], the transmuted Weibull-Exponetial distribution [38], the transmuted Lindley-Exponential distribution [39] and the transmuted Weibull-Rayleigh distribution [40].

Therefore this article aims at modifying the power Gompertz distribution [35] by introducing a new extension of the Gompertz distribution (transmuted power Gompertz distribution) using the method of quadratic rank transmutation map [8] considered previously by the above listed authors and hope that it will result to more better model compared to the power Gompertz distribution and will be useful for analyzing real life situations.

According to [35], the cumulative distribution function (cdf) of the Power Gompertz distribution (PGD) with parameters α , β *and* θ and its probability density function (pdf) are given as:

$$F\left(x\right)=1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \tag{1}$$

and

$$f\left(x\right)=\alpha\theta x^{\theta-1}\mathrm{e}^{\beta x^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \tag{2}$$

respectively. For $x > 0, \alpha > 0, \beta > 0, \theta > 0$ where α *and* β are scale and shape parameters of the model respectively and θ is the power parameter.

The remaining parts of this article are presented in sections as follows: the new distribution is defined with its graphical analysis in section 2. Section 3 derived some properties of the new distribution. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. An application of the new model with other existing distributions to two datasets is done in section 5 and a useful summary and conclusion is given in section 6.

2 The Transmuted Power Gompertz Distribution (TPGD)

According to [8], the cdf and pdf of the quadratic rank transmutation map are respectively given by:

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2 \tag{3}$$

and

$$f(x) = g(x)\left[1 + \lambda - 2\lambda G(x)\right] \tag{4}$$

where; x>0, and $-1 \le \lambda \le 1$ is the transmuted parameter, *G(x)* and g(x) are the cdf and pdf of the continuous distribution to be extended respectively.

Using the above method by [8], the cdf and pdf of the transmuted power Gompertz distribution are defined for a random variable *X* by substituting equation (1) and (2) in (3) and (4) and simplifying as:

$$F\left(x\right) = \left(1+\lambda\right)\left(1 - \mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)}\right) - \lambda\left[1 - \mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)}\right]^2$$

$$F\left(x\right) = 1 - \left(1-\lambda\right)\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)} - \lambda\mathrm{e}^{-\frac{2\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)} \tag{5}$$

and

$$f(x) = \alpha\theta x^{\theta-1}\mathrm{e}^{\beta x^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)}\left[1 + \lambda - 2\lambda\left(1 - \mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)}\right)\right]$$

$$f(x) = \alpha\theta x^{\theta-1}\mathrm{e}^{\beta x^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}} - 1\right)}\left[1 - \lambda + 2\lambda\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}} - 1\right)}\right] \tag{6}$$

respectively. Where, $x > 0, \alpha > 0, \theta > 0, -1 \le \lambda \le 1$ in which α is the shape parameter respectively and θ is the exponential parameter while λ is called the transmuted parameter.

The *pdf* and *cdf* of the TPGD using some parameter values are displayed in Fig. 1 as follows:

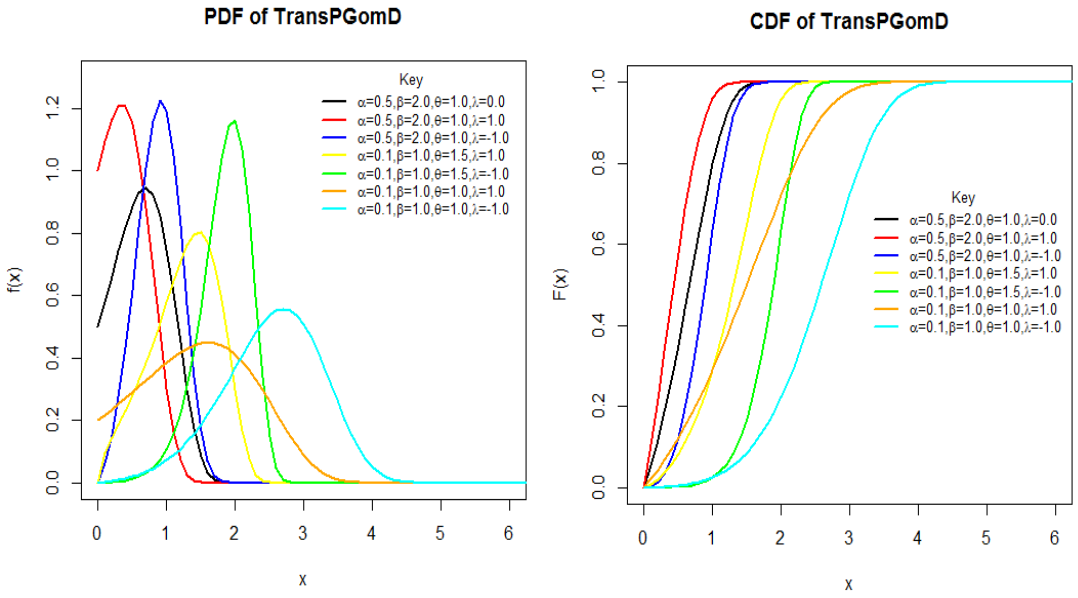

Fig. 1. *PDF* and CDF of the TransPGomD for different parameter values

Fig. 1 indicates that the TransPGomD distribution is positively skewed and takes various shapes depending on the parameter values. Also, from the above *cdf* plot, it is clear that the *cdf* approaches one (1) when *X* tends to infinity and equals zero when *X* tends to zero as normally expected.

3 Statistical Properties of TPGD

In this section, we derived, study and discuss some properties of the TransPGomD distribution. They are as follows:

3.1 Quantile function

Hyndman and Fan [41] defined the quantile function for any distribution in the form $Q(u) = X_q = F^{-1}(u)$ where $Q(u)$ is the quantile function of the random variable *X* for the cdf *F*(*x*) and u is uniform with the interval $0 < u < 1$.

Taking *F(x)* to be the *cdf* of the TransPGomD and inverting it as defined above will give us the quantile function as follows:

$$F(x) = 1 - (1-\lambda)\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} - \lambda \mathrm{e}^{-\frac{2\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} = u \tag{7}$$

Simplifying (7) above and solving for *X* presents the quantile function of the TransPGomD as:

$$Q(u) = \left\{\frac{1}{\beta}\log\left(1 - \frac{\beta(1-u)}{3\alpha}\right)\right\}^{\frac{1}{\theta}} \tag{8}$$

where *u* is a uniform variate on the unit interval (0,1). This function is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question as shown below.

Using (8) above, the median of *X* from the TransPGomD is simply obtained by setting *u*=0.5 and this substitution of $u = 0.5$ in (8) gives:

$$Median = \left\{ \frac{1}{\beta} \log\left(1 - \frac{\beta(0.5)}{3\alpha}\right)\right\}^{\frac{1}{\theta}} \tag{9}$$

Similarly, random numbers can be simulated from the TransPGomD by setting $Q(u) = X$ and this process is called simulation using inverse transformation method. This means for any $\alpha, \beta, \theta > 0$ and $u \in (0,1)$:

$$X = \left\{ \frac{1}{\beta} \log\left(1 - \frac{\beta(1-u)}{3\alpha}\right)\right\}^{\frac{1}{\theta}} \tag{10}$$

"where *u* is a uniform variate on the unit interval *(0,1)* and $W_{-1}(.)$ represents the negative branch of the Lambert function".

Again using the function above, the quantile based measures of skewness and kurtosis are obtained as follows:

Kennedy and Keeping [42] defined the Bowley's measure of skewness based on quartiles as:

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{11}$$

And Moors [43] presented the Moors' kurtosis based on octiles by:

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{8}\right)} \tag{12}$$

"where $Q(.)$ is calculated by using the quantile function from equation (8).

3.2 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left(X^n\right) = \int_0^{\infty} x^n f(x) dx \tag{13}$$

where *f(x)* is the *pdf* of the TransPGomD given in equation (6) as:

$$f(x)=\alpha\theta x^{\theta-1}e^{\beta x^{\theta}}e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\left[1-\lambda+2\lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]$$

Before substitution in (8), we perform the expansion and simplification of the *pdf* as follows:

$$\begin{aligned}f(x)&=\alpha(1-\lambda)\theta x^{\theta-1}e^{\beta x^{\theta}}e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}+2\alpha\lambda\theta x^{\theta-1}e^{\beta x^{\theta}}e^{-\frac{2\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\\ f(x)&=\alpha(1-\lambda)\theta x^{\theta-1}e^{\beta x^{\theta}}e^{\frac{\alpha}{\beta}\left(1-e^{\beta x^{\theta}}\right)}+2\alpha\lambda\theta x^{\theta-1}e^{\beta x^{\theta}}e^{\frac{2\alpha}{\beta}\left(1-e^{\beta x^{\theta}}\right)}\end{aligned} \tag{14}$$

First, by expanding the exponential term in (14) using power series, we obtain:

$$e^{\frac{\alpha}{\beta}\left(1-e^{\beta x^{\theta}}\right)}=\sum_{k=0}^{\infty}\frac{\alpha^k}{\beta^k k!}\left(1-e^{\beta x^{\theta}}\right)^k \tag{15}$$

and

$$e^{\frac{2\alpha}{\beta}\left(1-e^{\beta x^{\theta}}\right)}=\sum_{k=0}^{\infty}\frac{2^k\alpha^k}{\beta^k k!}\left(1-e^{\beta x^{\theta}}\right)^k \tag{16}$$

Making use of the result in (15) and (16) above, (14) becomes

$$\begin{aligned}f(x)&=\alpha(1-\lambda)\theta\sum_{k=0}^{\infty}\frac{\alpha^k}{\beta^k k!}\left(1-e^{\beta x^{\theta}}\right)^k x^{\theta-1}e^{\beta x^{\theta}}+2\alpha\lambda\theta\sum_{k=0}^{\infty}\frac{2^k\alpha^k}{\beta^k k!}\left(1-e^{\beta x^{\theta}}\right)^k x^{\theta-1}e^{\beta x^{\theta}}\\ f(x)&=\alpha(1-\lambda)\theta\sum_{k=0}^{\infty}\frac{\alpha^k}{\beta^k k!}x^{\theta-1}e^{\beta x^{\theta}}\left(1-e^{\beta x^{\theta}}\right)^k+2\alpha\lambda\theta\sum_{k=0}^{\infty}\frac{2^k\alpha^k}{\beta^k k!}x^{\theta-1}e^{\beta x^{\theta}}\left(1-e^{\beta x^{\theta}}\right)^k\end{aligned} \tag{17}$$

Also, using the generalized binomial theorem, we can write the last term from the above result as:

$$\left(1-e^{\beta x^{\theta}}\right)^k=\sum_{l=0}^{\infty}(-1)^l\binom{k}{l}e^{l\beta x^{\theta}} \tag{18}$$

Making use of the result in (18) above in (17) and simplifying, we obtain:

$$f(x)=\alpha(1-\lambda)\theta\sum_{k=0}^{\infty}\frac{\alpha^k}{\beta^k k!}x^{\theta-1}e^{\beta x^{\theta}}\sum_{l=0}^{\infty}(-1)^l\binom{k}{l}e^{l\beta x^{\theta}}+2\alpha\lambda\theta\sum_{k=0}^{\infty}\frac{2^k\alpha^k}{\beta^k k!}x^{\theta-1}e^{\beta x^{\theta}}\sum_{l=0}^{\infty}(-1)^l\binom{k}{l}e^{l\beta x^{\theta}}$$

$$f(x)=\alpha(1-\lambda)\theta\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k(-1)^l}{\beta^k k!}\binom{k}{l}x^{\theta-1}e^{\beta(1+l)x^{\theta}}+2\alpha\lambda\theta\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k 2^k(-1)^l}{\beta^k k!}\binom{k}{l}x^{\theta-1}e^{\beta(1+l)x^{\theta}}$$

$$f(x)=\left[\frac{(1-\lambda)}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k(-1)^l}{\beta^k k!}\binom{k}{l}+\frac{2\lambda}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k 2^k(-1)^l}{\beta^k k!}\binom{k}{l}\right]x^{\theta-1}e^{\beta(1+l)x^{\theta}} \tag{19}$$

Now, using the simplified pdf of the TransPGomD in equation (19), the n[th] ordinary moment of the TransPGomD is derived as follows:

$$\mu_n^{'} = E\left(X^n\right) = \int_0^{\infty} x^n f(x)dx = \left[\frac{(1-\lambda)}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k(-1)^l}{\beta^k k!}\binom{k}{l} + \frac{2\lambda}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k 2^k(-1)^l}{\beta^k k!}\binom{k}{l}\right]\int_0^{\infty} x^{n+\theta-1}\mathrm{e}^{\beta(1+l)x^{\theta}}dx \tag{20}$$

Making use of integration by substitution method in equation (20), we perform the following operations:

Let $-u = \beta(1+l)x^{\theta} \Rightarrow x = u^{\frac{1}{\theta}}\left(\frac{-1}{\beta(1+l)}\right)^{\frac{1}{\theta}}$ which implies that $-\frac{du}{dx} = \theta\beta(1+l)x^{\theta-1} \Rightarrow dx = -\frac{x^{-(\theta-1)}du}{\theta\beta(1+l)}$.

Substituting for $x,$ and dx in equation (20) and simplifying; we have:

$$\mu_n^{'} = \left[\frac{(1-\lambda)}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k(-1)^l}{\beta^k k!}\binom{k}{l} + \frac{2\lambda}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k 2^k(-1)^l}{\beta^k k!}\binom{k}{l}\right]\frac{1}{\theta}\left(\frac{-1}{\beta(1+l)}\right)^{\frac{n}{\theta}+1}\int_0^{\infty} u^{\frac{n}{\theta}+1-1}\mathrm{e}^{-u}du \tag{21}$$

Using the definition of complete Gamma function, we obtain the n^{th} ordinary moment of X for the TransPGomD as:

$$\mu_n^{'} = \left[\frac{(1-\lambda)}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k(-1)^l}{\beta^k k!}\binom{k}{l} + \frac{2\lambda}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k 2^k(-1)^l}{\beta^k k!}\binom{k}{l}\right]\frac{\Gamma\left(\frac{n}{\theta}+1\right)}{\theta}\left(\frac{-1}{\beta(1+l)}\right)^{\frac{n}{\theta}+1} \tag{22}$$

The mean ($\mu_1^{'}$), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS), coefficient of kurtosis (CK), moment generating function (mgf) and characteristics function (cf) can all be calculated based on the ordinary moments in equation (22) using some simple and well-defined relationships (see details in [44]).

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{23}$$

Applying power series expansion and simplifying (23) gives the following:

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\mu_r^{'}\right] \tag{24}$$

Using the result in (24) and simplifying the integral in (23) therefore we have:

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left\{\left[\frac{(1-\lambda)}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k(-1)^l}{\beta^k k!}\binom{k}{l} + \frac{2\lambda}{(\alpha\theta)^{-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^k 2^k(-1)^l}{\beta^k k!}\binom{k}{l}\right]\frac{\Gamma\left(\frac{r}{\theta}+1\right)}{\theta}\left(\frac{-1}{\beta(1+l)}\right)^{\frac{r}{\theta}+1}\right\} \tag{25}$$

The characteristics function of a random variable X is defined by:

$$\varphi_x(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x)\,dx \tag{26}$$

Again, applying power series expansion and simplifying equation (26), we obtained the characteristics function of X as:

$$\varphi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{ \left[\frac{(1-\lambda)}{(\alpha\theta)^{-1}} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha^k (-1)^l}{\beta^k k!} \binom{k}{l} + \frac{2\lambda}{(\alpha\theta)^{-1}} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha^k 2^k (-1)^l}{\beta^k k!} \binom{k}{l} \right] \frac{\Gamma\left(\frac{r}{\theta}+1\right)}{\theta} \left(\frac{-1}{\beta(1+l)} \right)^{\frac{r}{\theta}+1} \right\} \tag{27}$$

The Cumulant generating function (CGF) is obtained as:

$$K_x(t) = \log\left\{ \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\left[\frac{(1-\lambda)}{(\alpha\theta)^{-1}} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha^k (-1)^l}{\beta^k k!} \binom{k}{l} + \frac{2\lambda}{(\alpha\theta)^{-1}} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha^k 2^k (-1)^l}{\beta^k k!} \binom{k}{l} \right] \frac{\Gamma\left(\frac{r}{\theta}+1\right)}{\theta} \left(\frac{-1}{\beta(1+l)} \right)^{\frac{r}{\theta}+1} \right) \right\} \tag{28}$$

3.3 Reliability analysis of the TransPGomD

The Survival function (SF) describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{29}$$

Applying the cdf of the TPGD in (29) and simplifying, the survival function for the TransPGomD is obtained as:

$$S(x) = e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \left[1 - \lambda + \lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \right] \tag{30}$$

The hazard function (HF) is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{31}$$

Meanwhile, the expression for the hazard rate of the TransPGomD is given by

$$h(x) = \frac{\alpha\theta x^{\theta-1} e^{\beta x^{\theta}} \left[1 - \lambda + 2\lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \right]}{\left[1 - \lambda + \lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \right]} \tag{32}$$

Hence, equation (30) and (32) are the survival and the hazard functions of the transmuted power Gompertz distribution (TransPGomD) respectively and their plots are given in the figure below.

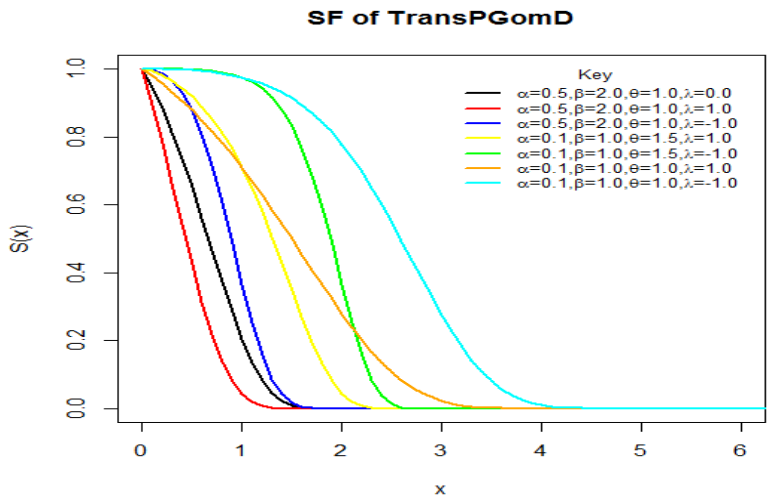

Fig. 2(a). Survival function (SF) of the TransPGomD for different parameter values

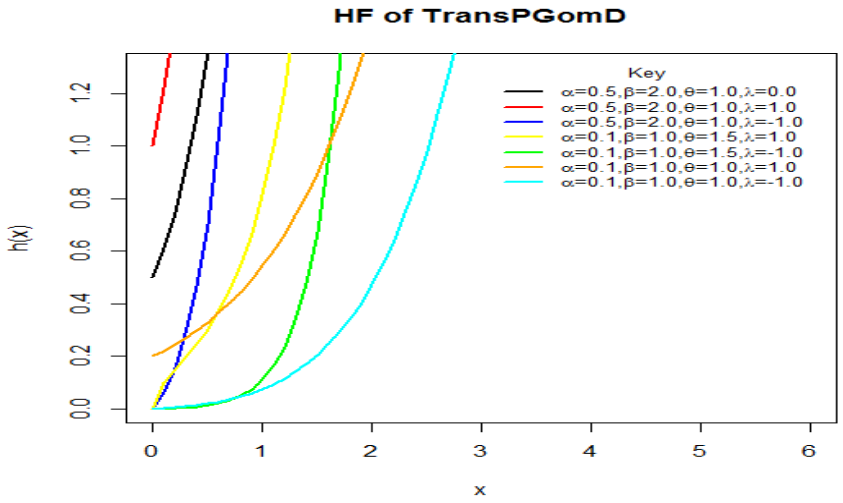

Fig. 2(b). Hazard function (HF) of the TransPGomD for different parameter values

The plots in Fig. 2(a) show that the probability of survival equals one (1) at initial time or early age and it decreases as time increases and equals zero (0) as time approaches infinity.

It is also shown in Fig. 2(b) that the TransPGomD has increasing failure rate which implies that the probability of failure for any random variable following a TransPGomD increases as time increases, that is, probability of failure or death increases as life ages.

3.4 Order statistics

Suppose $X_1, X_2, ..., X_n$ is a random sample from the TransPGomD (TPGomD) and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}f(x)F(x)^{k+i-1} \tag{33}$$

Using (5) and (6), the pdf of the i^{th} order statistics $X_{(i:n)}$, can be expressed from (33) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[\frac{\alpha\theta x^{\theta-1}e^{\beta x^{\theta}}}{e^{\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}\left[1-\lambda+2\lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]\right]\left[1-\frac{(1-\lambda)}{e^{\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}-\frac{\lambda}{e^{\frac{2\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}\right]^{i+k-1} \tag{34}$$

Hence, the pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the TransPGomD are respectively given by;

$$f_{1:n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha\theta x^{\theta-1}e^{\beta x^{\theta}}}{e^{\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}\left[1-\lambda+2\lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]\right]\left[1-\frac{(1-\lambda)}{e^{\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}-\frac{\lambda}{e^{\frac{2\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}\right]^{k} \tag{35}$$

and

$$f_{n:n}(x)=n\left[\frac{\alpha\theta x^{\theta-1}e^{\beta x^{\theta}}}{e^{\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}\left[1-\lambda+2\lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]\right]\left[1-\frac{(1-\lambda)}{e^{\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}-\frac{\lambda}{e^{\frac{2\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}\right]^{n-1} \tag{36}$$

4 Estimation of Unknown Parameters of the TransPGomD

In this section, the estimation of the parameters of the TransPGomD is done by using the method of maximum likelihood estimation (MLE). Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the TransPGomD with unknown parameters α, β, θ and λ all defined previously in section one.

The likelihood function of the TransPGomD using the pdf in equation (6) is given by;

$$L\left(\underline{X}/\alpha,\beta,\theta,\lambda\right)=(\alpha\theta)^n e^{\beta\sum_{i=1}^{n}x_i^{\theta}}e^{-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i^{\theta}}-1\right)}\prod_{i=1}^{n}\left\{x_i^{\theta-1}\left[1-\lambda+2\lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}}-1\right)}\right]\right\} \tag{37}$$

Let the natural logarithm of the likelihood function be, $l(\eta)=\log L\left(\underline{X}\mid\alpha,\beta,\theta,\lambda\right)$, therefore, taking the natural logarithm of the function above gives:

$$l(\eta)=n\log\alpha+n\log\theta+(\theta-1)\sum_{i=1}^{n}\log x+\beta\sum_{i=1}^{n}x_i^{\theta}-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i^{\theta}}-1\right)+\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda e^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}}-1\right)}\right] \tag{38}$$

Differentiating $l(\eta)$ partially with respect to parameters α, β, θ and λ respectively gives the following results;

$$\frac{\partial l(\eta)}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i^{\theta}} - 1\right) - \frac{2\lambda}{\beta}\sum_{i=1}^{n}\left\{\frac{\left(\mathrm{e}^{\beta x_i^{\theta}} - 1\right)\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)}}{1-\lambda+2\lambda\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)}}\right\} \tag{39}$$

$$\frac{\partial l(\eta)}{\partial \beta} = \sum_{i=1}^{n} x_i^{\theta} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{x_i^{\theta} e^{\beta x_i^{\theta}} - \frac{\alpha}{\beta^2}\left(e^{\beta x_i^{\theta}} - 1\right)\right\} - \frac{2\alpha\lambda}{\beta}\sum_{i=1}^{n}\left\{\frac{\left[x_i^{\theta} e^{\beta x_i^{\theta}} - \frac{\alpha}{\beta^2}\left(e^{\beta x_i^{\theta}} - 1\right)\right]\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)}}{1-\lambda+2\lambda\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)}}\right\} \tag{40}$$

$$\frac{\partial l(\eta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x + \beta\sum_{i=1}^{n} x_i^{\theta}\ln x_i - \alpha\sum_{i=1}^{n} x_i^{\theta}\ln x_i e^{\beta x_i^{\theta}} - 2\alpha\lambda\sum_{i=1}^{n}\left\{\frac{x_i^{\theta}\ln x_i e^{\beta x_i^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)}}{1-\lambda+2\lambda\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)}}\right\} \tag{41}$$

$$\frac{\partial l(\eta)}{\partial \lambda} = \sum_{i=1}^{n}\left\{\frac{2\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)} - 1}{1-\lambda+2\lambda\mathrm{e}^{-\frac{\alpha}{\beta}\left(e^{\beta x_i^{\theta}} - 1\right)}}\right\} \tag{42}$$

Making equation (39), (40), (41) and (42) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters $\hat{\alpha}, \hat{\beta}$, $\hat{\theta}$ and $\hat{\lambda}$. However, these solutions cannot be obtained manually except numerically with the aid of suitable statistical software like *R*, *SAS*, *MATHEMATICA* e.t.c. Hence, some datasets are being considered in the next section to fit the new distribution with other related distributions using "maxLik" package in R software.

5 Applications to Three Real Life Datasets

This section presents two real life datasets, their descriptive statistics, graphical summary and some distributions fitted to the datasets. The section presents and compares the fits of the transmuted Power Gompertz Distribution (TPGD), transmuted Gompertz distribution (TGD), Power Gompertz Distribution (PGD) and the Gompertz Distribution (GD) based on the two datasets mentioned above.

To check the fitness and flexibility of the models listed above, the Akaike Information Criterion (AIC) is chosen and used for selecting the best model. The formula for this criterion is given as:

$$AIC = -2\ell + 2k$$

Where ℓ denotes the value of log-likelihood function evaluated at the *MLEs*, *k* is the number of model parameters and *n* is the sample size. Meanwhile, when taking our decisions the model with the lowest values of AIC would be chosen as the best model to fit the data.

Tables 2 and 4 list the Maximum Likelihood Estimates of the model parameters and the AIC values for the fitted models based on the first and second datasets respectively.

Data set I: This data represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929-1938 from [45,46] and [47]. Its summary is given as follows:

Table 1. Summary statistics for the dataset I

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
121	0.30	17.5	40.00	60.0	46.33	154.00	1244.464	1.04318	0.40214

From the descriptive statistics in table 1, it is seen that dataset I is positively skewed and not normally distributed.

Table 2. Performance of the distributions using the value of AIC based on dataset I

Distributions	Parameter estimates	log-likelihood value	*AIC*	Rank of models
TPGomD	$\hat{\alpha}$ =0.09434	-579.3027	1166.605	**1st**
	$\hat{\beta}$ =0.36095			
	$\hat{\theta}$ =0.42241			
	$\hat{\lambda}$ =-0.96519			
PGomD	$\hat{\alpha}$ = 0.020775	-581.2547	1168.509	**3rd**
	$\hat{\beta}$ = 0.021019			
	$\hat{\theta}$ = 0.870416			
TGomD	$\hat{\alpha}$ = 0.011294	-580.4765	1166.953	**2nd**
	$\hat{\beta}$ = 0.008866			
	$\hat{\lambda}$ = 0.297067			
GomD	$\hat{\alpha}$ = 0.018400	-583.1125	1170.225	**4th**
	$\hat{\beta}$ = 0.007703			

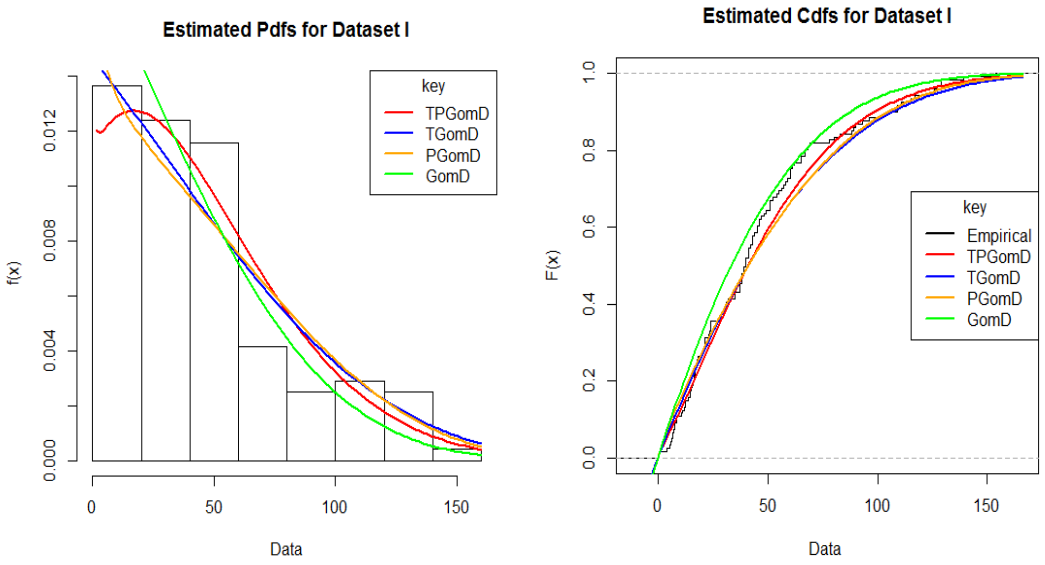

Fig. 3. Histogram and estimated densities (Pdfs) and cdfs of the models fitted to dataset I

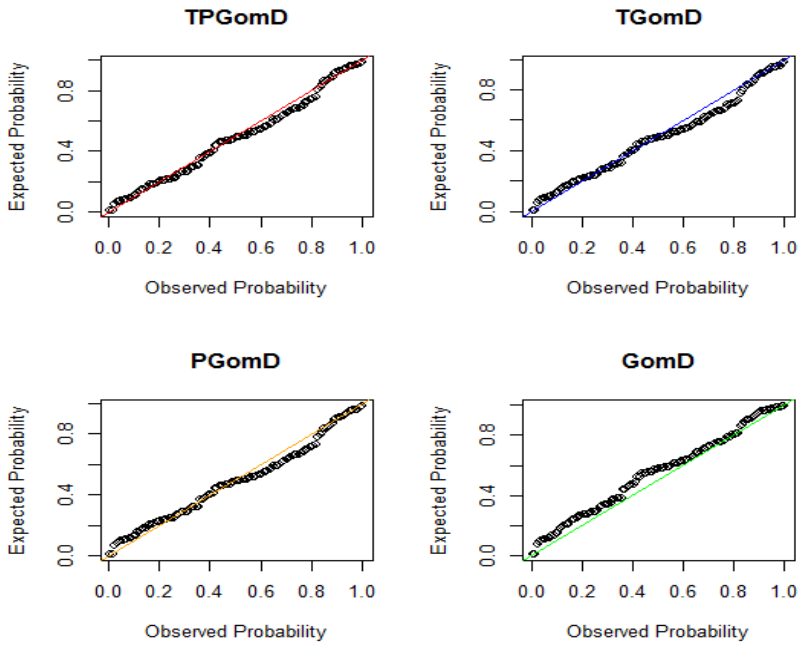

Fig. 4. Probability plots for the fit of the TPGomD, TGomD, PGomD and GomD based on dataset I

Data set II: This dataset represents the survival times (in days) of 72 guinea pigs infected with virulenttubercle bacilli reported by [48] and also used by [39]. The summary is presented in Table 3 below.

Table 3. Descriptive statistics for dataset II

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
72	10.0	108.0	149.5	224.0	176.8	555.0	10705.1	1.34128	1.98852

From the descriptive statistics in Table 3, it can be observed that dataset II is also positively skewed.

Table 4. Performance of the distributions using the value of AIC based on dataset II

Distributions	Parameter estimates	log-likelihood value	*AIC*	Rank of models
TPGomD	$\hat{\alpha}$ =0.18958	-390.8596	789.7193	**1st**
	$\hat{\beta}$ =0.37333			
	$\hat{\theta}$=0.28835			
	$\hat{\lambda}$ =-5.80601			
PGomD	$\hat{\alpha}$ =0.006765	-433.2963	872.5925	**2nd**
	$\hat{\beta}$ = 0.902166			
	$\hat{\theta}$ = 0.319790			
TGomD	$\hat{\alpha}$ =0.0032698	-433.3389	872.6778	**3rd**
	$\hat{\beta}$ = -0.0038057			
	$\hat{\lambda}$ =3.1017073			
GomD	$\hat{\alpha}$ =0.0034090	-438.6373	881.2746	**4th**
	$\hat{\beta}$ = 0.0039771			

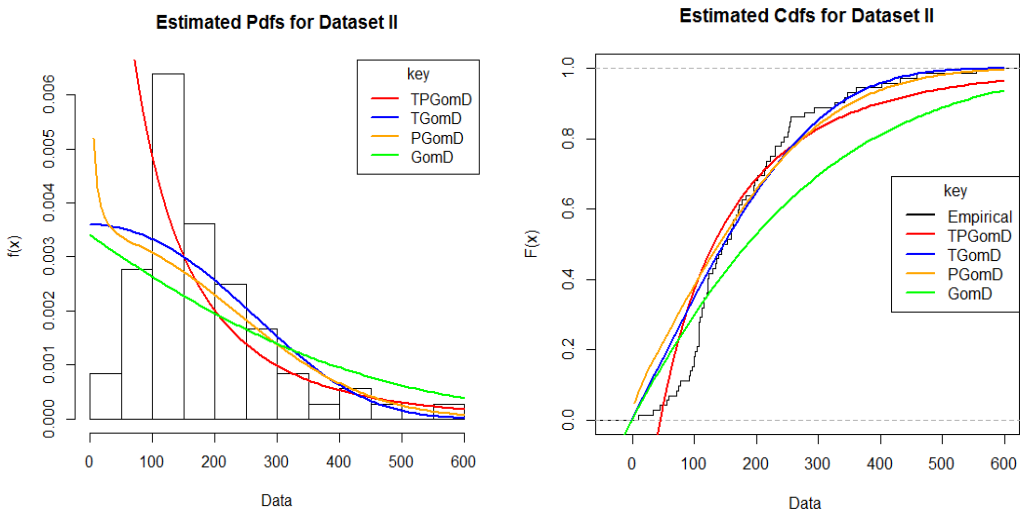

Fig. 5. Histogram and estimated densities (pdfs) and cdfs of the models fitted to dataset II

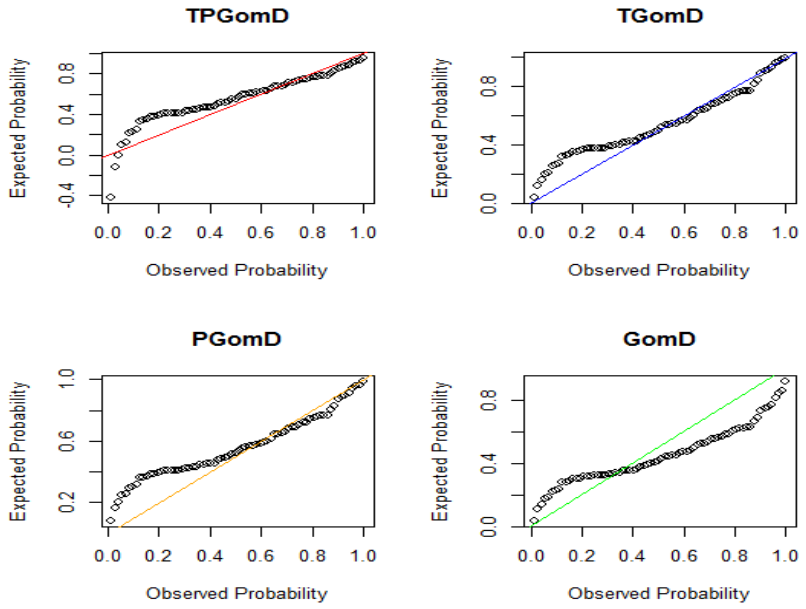

Fig. 6. Probability plots for the fit of the TPGomD, TGomD, PGomD and GomD based on dataset II

Tables 2 and 4 present the parameter estimates and the values of AIC for the four fitted distributions using dataset I and dataset II respectively. Based on the values of AIC in Table 2 and those in Table 4, it can be seen that AIC values of TPGD are smaller compared to those of TGomD, PGomD and the GomD and these results indicate that the transmuted Power Gompertz distribution (TPGomD) fits the two datasets better than the other three distributions (TGomD, PGomD and the GomD). This result and performance is confirmed from the estimated pdfs and cdfs in Fig. 3 and 5 as well as the probability plots presented in Figs. 4 and 6. The result is an affirmation to the fact that the method quadratic rank transmutation map for adding a single or transmuted parameter has increased the flexibility of the power Gompertz distribution. Hence, this study is in line with the conclusions in [33,36,37,38, 39,40,49,50,51].

6 Summary and Conclusion

In conclusion, the present article has used the method of quadratic rank transmutation map to defined a transmuted power Gompertz distribution and study its properties such as quantile function, moments, moment generating function, characteristics function, survival function, hazard function, distribution of order statistics among others. The research has discussed analytically and in practice the usefulness and applications of these properties. The study has also demonstrated earlier in the preceding section with applications of the model to real life datasets that the new distribution (TPGomD) has a better fit to the two datasets compared to the transmuted Gompertz, Power Gompertz and the conventional Gompertz distributions.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Johnson NL, Kotz S, & Balakrishnan N. Continuous Univariate Distributions, 2 edn, John Wiley and Sons, New York. 1995;2.

[2] Brown K, Forbes W. A mathematical model of aging processes. Journal of Gerontology. 1974;29(1): 46–51.

[3] Ohishi K, Okamura H, Dohi T. Gompertz software reliability model: Estimation algorithm and empirical validation. J. of Syst. and Software. 2009;82(3):535-543.

[4] Economos AC. Rate of aging, rate of dying and the mechanism of mortality. Archives of Gerontology and Geriatrics. 1982;1(1):46–51.

[5] Bemmaor AC, Glady N. Modeling purchasing behavior with sudden death: A flexible customer lifetime model. Management Science. 2012;58(5):1012–1021.

[6] Willemse W, Koppelaar H. Knowledge elicitation of Gompertz' law of mortality, Scandinavian Actuarial Journal. 2000;2:168–179.

[7] Eugene N, Lee C, Famoye F. Beta-normal distribution and it applications. Comm. in Stat.: Theo. and Meth. 2002;31:497-512.

[8] Shaw WT, Buckley IR. The alchemy of probability distributions: Beyond Gram-Charlier expansions and a Skew-Kurtotic-normal distribution from a rank transmutation map. Research Report; 2007.

[9] Zografos K, Balakrishnan N. On families of beta- and generalized gamma generated distributions and associated inference. Stat. Meth. 2009;63:344-362.

[10] Cordeiro GM, De Castro M. A new family of generalized distributions. J. Stat. Comp. Sim. 2011;81: 883-898.

[11] Alexander C, Cordeiro GM, Ortega EMM, Sarabia JM. Generalized beta-generated distributions. Comp. Stat. Data Analy. 2012;56:1880-1897.

[12] Risti´c MM, Balakrishnan N. The gamma-exponentiated exponential distribution. J. Stat. Comp. Sim. 2012;82:1191-1206.

[13] Torabi H, Montazari NH. The gamma-uniform distribution and its application. Kybernetika. 2012;48: 16-30.

[14] Amini M, Mirmostafaee SMTK, Ahmadi J. Log-gamma-generated families of distributions. Stat.; 2012.
DOI: 10.1008/02331888.2012.748775

[15] Alzaghal A, Lee C, Famoye F. Exponentiated T-X family of distributions with some applications. Int. J. Prob. Stat. 2013;2:31-49.

[16] Cordeiro GM, Ortega EMM, da Cunha DCC. The exponentiated generalized class of distribution. J. of Data Sci. 2013;11:1-27.

[17] Alzaatreh A, Famoye F, Lee C. Weibull-Pareto distribution and its applications. Comm. Stat. Theo. Meth. 2013;42:1673-1691.

[18] Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G Family of Probability distributions. J. Data Sci. 2014;12:53-68.

[19] Torabi H, Montazari NH. The logistic-uniform distribution and its application. Comm. Stat.: Sim. and Comp. 2014;43:2551-2569.

[20] Alzaatreh A, Famoye F, Lee C. The gamma-normal distribution: Properties and applications. Comp. Stat. Data Analy. 2014;69:67-80.

[21] Cordeiro GM, Ortega EMM, Popovic BV, Pescim RR. The Lomax generator of distributions: Properties, minification process and regression model. Appl. Math. Comp. 2014;247:465-486.

[22] Cordeiro GM, Ortega EMM, Ramires TG. A new generalized Weibull family of distributions: Mathematical properties and applications. J. Stat. Distr. Appl. 2015;2:13.
DOI: 10.1186/s40488-015-0036-6

[23] Alizadeh M, Cordeiro GM, de Brito E, Demetrio CGB. The beta Marshall-Olkin family of distributions. J. of Stat. Dist. Appl. 2015;2(4):1-18.

[24] Tahir MH, Cordeiro GM, Alzaatreh A, Mansoor M, Zubair M. The logistic-X family of distributions and its applications. Comm. Stat.: Th. and Meth. 2016;45(24):7326-7349.

[25] Tahir MH, Zubair M, Mansoor M, Cordeiro GM, Alizadeh M. A new Weibull-G family of distributions. Hac. J. Math. Stat. 2016;45(2):629-647.

[26] Cakmakyapan S, Ozel G. The Lindley family of distributions: Properties and applications. Hacettepe Journal of Mathematics and Statistics. 2016;46:1-27.
DOI: 10.15672/hjms.201611615850

[27] Alizadeh M, Cordeiro GM, Pinho BLG, Ghosh I. The Gompertz-G family of distributions. J. of Stat. Theo. and Pract. 2017;11(1):179-207.
DOI: 10.1080/15598608.2016.1267668

[28] Gomes-Silva F, Percontini A, De Brito E, Ramos MW, Venancio R, Cordeiro GM. The odd Lindley-G family of distributions. Austrian J. of Stat. 2017;46:65-87.

[29] El-Gohary A, Al-Otaibi AN. The generalized Gompertz distribution. Applied Mathematical Modelling. 2013;37(1-2):13-24.

[30] Gupta RD, Kundu D. Generalized exponential distribution. Australian and New Zealand Journal of Statistics. 1999;41:173-188.

[31] Jafari1 AA, Tahmasebi S, Alizadeh M. The beta-gompertz distribution. Revista Colombiana de Estadística Junio. 2014;37(1):139-156.

[32] El-Damcese MA, Mustafa A, El-Desouky BS, Mustafa ME. The odd generalized exponential gompertz distribution. Applied Mathematics. 2015;6:2340-2353.
Available:http://dx.doi.org/10.4236/am.2015.614206

[33] Abdul-Moniem IB, Seham M. Transmuted gompertz distribution. Computational and Applied Mathematics Journal. 2015;1(3):88-96.

[34] Omale A, Yahaya A, Asiribo OE. On properties and applications of lomax-gompertz distribution. Asian J. of Prob. and Stat. 2019;3(2):1-17.
DOI: 10.9734/AJPAS/2019/v3i230092

[35] Ieren TG, Kromtit FM, Agbor BU, Eraikhuemen IB, Koleoso PO. A power gompertz distribution: Model, properties and application to bladder cancer data. Asian Res. J. of Math. 2019;15(2):1-14.

[36] Abdullahi UK, Ieren TG. On the inferences and applications of transmuted exponential Lomax distribution. Int. J. of Adv. Prob. and Stat. 2018;6(1):30-36.

[37] Abdullahi J, Abdullahi UK, Ieren TG, Kuhe DA, Umar AA. On the properties and applications of transmuted odd generalized exponential- exponential distribution. Asian J. of Prob. and Stat. 2018; 1(4):1-14.

[38] Yahaya A, Ieren TG. On transmuted weibull-exponential distribution: Its properties and applications. Nigerian J. of Sci. Res. 2017;16(3):289-297.

[39] Umar AA, Eraikhuemen IB, Koleoso PO, Joel J, Ieren TG. On the properties and applications of a transmuted lindley-exponential distribution. Asian J. of Prob. & Stat. 2019;5(3):1-13.

[40] Yahaya A, Abdullahi J, Ieren TG. Properties and applications of a transmuted Weibull–Rayleigh distribution. Science Forum (Journal of Pure and Applied Sciences). 2020;19:126-138.

[41] Hyndman RJ, Fan Y. Sample quantiles in statistical packages. The American Stat. 1996;50(4):361-365.

[42] Kenney JF, Keeping ES. Mathematics of statistics. 3 edn, Chapman & Hall Ltd, New Jersey; 1962.

[43] Moors JJ. A quantile alternative for kurtosis. J. of the Royal Stat. Society: Series D. 1988;37:25-32.

[44] Eraikhuemen IB, Ieren TG, Mabur TM, Sa'ad M, Kuje S, Chama AF. A study of properties and applications of a lomax gompertz-makeham distribution. Asian Res. J. of Math. 2019;15(3):1-27.

[45] Lee ET. Statistical methods for survival data analysis. 2nd Edn., John Wiley and Sons Ltd., New York, USA. 1992;496.
[ISBN-13: 9780471615927]

[46] Ramos MWA, Cordeiro GM, Marinho PRD, Dias CRB, Hamedani GG. The zografos-balakrishnan log-logistic distribution: Properties and applications. J. Stat. Theory Applic. 2013;12:225-244.

[47] Oguntunde PE, Adejumo AO, Owoloko EA. Exponential inverse exponential (EIE) distribution with applications to lifetime data. Asian J. Sci. Res. 2017;10:169-177.

[48] Bjerkedal T, Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli, Am. J. Hyg. 1960;72:130-148.

[49] Owoloko EA, oguntunde PE, Adejumo AO. Performance rating of the transmuted exponential distribution: An analytical approach. Spring. 2015;4:818-829.

[50] Merovci F. The transmuted rayleigh distribution. Austr. J. Stat. 2013;22(1):21-30.

[51] Merovci F, Puka I. Transmuted pareto distribution. Prob. Stat. Forum. 2014;7:1-11.

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