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Exact Explicit Solutions of a Nonlinear Evolution Equation

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Abstract

We employ bifurcation method of dynamical systems to investigate exact traveling wave solutions of a nonlinear evolution equation. We obtain some exact explicit expressions of solitary wave solutions and some new exact periodic wave solutions in parameter forms of Jacobian elliptic function. We point out that the solitary waves are limits of the periodic waves in some sense, the results infer that the periodic waves degenerate solitary waves in some conditions.

Keywords: Bifurcation method; solitary wave solutions; periodic waves solution.
2010 Mathematics Subject Classification: 34A34; 34A45; 35B20; 58B05; 74J35

1 Introduction

The Benjamin-Bona-Mahony(BBM) equation [1]

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1}$$

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was derived to describe propagation of long waves where nonlinear dispersion is incorporated. The spatially one-dimensional KdV equation

$$u_t + auu_x + u_{xxx} = 0, \tag{2}$$

is a approximate model that governs the one-dimensional propagation of small amplitude, weakly dispersive waves, and plays a major role in the solitons concepts. The term soliton coined by Zabusky and Kruskal [2] who found particle like waves which retained their shapes and velocities after collisions. The balance between the nonlinear convection term uu_x and the dispersion effect term u_{xxx} in the KdV equation (2) gives rise to solitons. Furthermore, both BBM and KdV equations can be used to describe long wave length in liquids, etc.
Besides, there are two well-known two-dimensional equations which were derived as the generalizations of the KdV equations. One is the the Kadomtsov-Petviashivilli(KP) equation [3],

$$(u_t + auu_x + u_{xxx})_x + u_{yy} = 0 \tag{3}$$

and the other is the Zakharov-Kuznetsov(ZK) equation [4]

$$u_t + auu_x + (u_{xx} + u_{yy} + u_{zz})_x = 0. \tag{4}$$

The studies made in the literature [5] dealed with the BBM equation and its modified forms formulated in the KP and ZK sense, and the BBM equation in KP sense was studied and some exact solutions were obtained [6],[7]. Further, to extend the relevant results, this work will investigate exact solutions of the nonlinear (2+1) dimensional ZK-BBM equation(5)

$$u_t + u_x - a(u^2)_x - (bu_{xt} + ku_{yt})_x = 0, \tag{5}$$

which is a generalized form of the ZK-BBM equation(6)

$$u_t + u_x + a(u^n)_x + b(u_{xx} + u_{yy}) = 0.(n > 1) \tag{6}$$

Some methods are applied to seek exact solutions of nonlinear evolution equations because exact solutions play a key role in comprehension of nonlinear phenomena. For example, the method of lines and Adomian decomposition is applied to obtain solitary wave solutions of the KdV equation [8]. Homotopy perturbation Pade technique is used to construct approximate and exact solutions of Boussinesq equations [9], extended tanh method, extended mapping method with symbol computation and bifurcation method of dynamical systems are used to study equation (5) [8],[9], and some solitary wave solutions and triangle periodic wave solutions were obtained.

However, there is no method can be used to all nonlinear evolution equations. The research on the solutions of the ZK-BBM equation now appears insufficient. Further studies are necessary for the traveling wave solutions of the ZK-BBM equation. The purpose of this paper is to apply the bifurcation method of dynamical systems [10],[11],[12],[13] to continue to seek traveling waves of equation (5). Firstly, we obtain some solitary wave solutions. Then, we get some new periodic wave solutions in parameter forms of Jacobian elliptic function. The periodic wave solutions obtained in this paper are new. Furthermore, we find an close relationship between the solitary waves and periodic waves, that is, the solitary waves are limits of the periodic waves in the sense of modulus of Jacobian elliptic function approaches 1.
This paper is organized as follows. In Sec. 2, we discuss the bifurcation phase portraits of planar system according to the ZK-BBM equation under different parameters conditions. In Sec.3, we give exact solitary wave solutions to the ZK-BBM equation. In Sec.4, we obtain periodic solitary waves in the forms of Jacobian elliptic function. Finally we discuss the relationship between the two kind waves in Sec. 5.

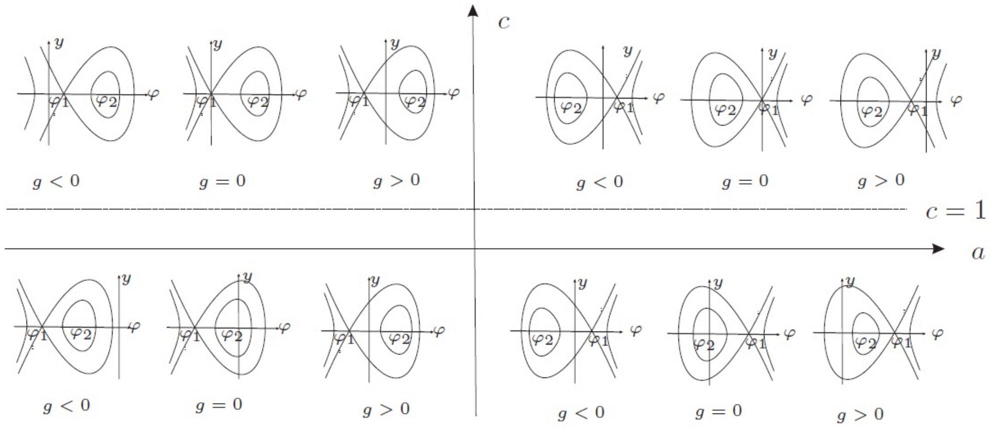

Figure 1: The phase portraits of system (9) under conditon $(b+c)k<0$

2 Bifurcation Phase Portraits

Let $u(x,y,t)=\varphi(\xi)$, $\xi=x+y-ct$, where c is the wave speed. Substituting $u(x,y,t)=\varphi(\xi)$ into (5) admits to the following ODE

$$(1-c)\varphi^{'}-a(\varphi^2)^{'}+(b+k)c\varphi^{'''}=0, \tag{7}$$

where the prime stands for the derivative with respect to ξ. Integrating (7) once with respect to ξ, it follows that

$$(1-c)\varphi-a\varphi^2+(b+k)c\varphi^{''}=g, \tag{8}$$

where g is the integral constant.
Equation (8) can be transformed to the following two-dimensional planar system

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y, \qquad \frac{\mathrm{d}y}{\mathrm{d}\xi}=\frac{a\varphi^2+(c-1)\varphi+g}{(b+k)c}. \tag{9}$$

System (9) has the first integral

$$H(\varphi,y)=\frac{a}{3(b+k)c}\varphi^3+\frac{c-1}{2(b+k)c}\varphi^2+\frac{g}{(b+k)c}\varphi-\frac{1}{2}y^2=h, \tag{10}$$

where h is the constant of integration.
Let $\Delta=(c-1)^2-4ag$. When $\Delta>0$, there are two equilibrium points $(\varphi_1,0)$ and $(\varphi_2,0)$ of (9) on $\varphi-$axis, where $\varphi_1=\frac{(1-c)+\sqrt{\Delta}}{2a}$, $\varphi_2=\frac{(1-c)-\sqrt{\Delta}}{2a}$. The Hamiltonian H of $(\varphi_1,0)$ and $(\varphi_2,0)$ is denoted by $h_1=H(\varphi_1,0)$ and $h_2=H(\varphi_2,0)$. According to the stationary theorem of differential equation, the bifurcation phase portraits of system (9) are given as Fig.1 and Fig.2 respectively in the case of $(b+k)c<0$ and $(b+k)c>0$, in which there are some homoclinic and periodic orbits.

3 Exact Explicit Expressions of Solitary Wave Solutions

In this Section, we solve solitary wave solutions under $g=0$ and $g\neq 0$ respectively.

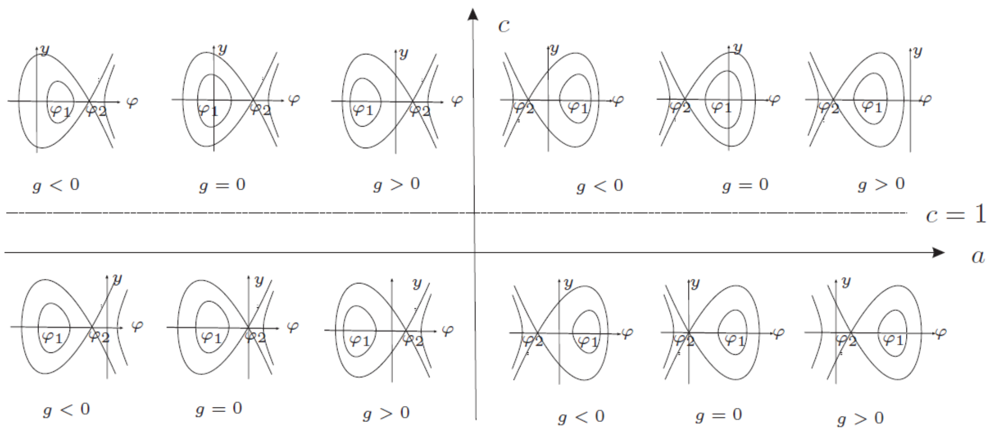

Figure 2: The phase portraits of system (9) under conditon $(b+c)k > 0$

3.1 The case integral constant $g = 0$

System (9) is namely the system as follows

$$\frac{d\varphi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{a\varphi^2 + (c-1)\varphi}{(b+k)c}, \tag{11}$$

which has the first integral

$$H(\varphi, y) = \frac{1}{2}y^2 - \frac{a}{3(b+k)c}\varphi^3 - \frac{c-1}{2(b+k)c}\varphi^2 = h. \tag{12}$$

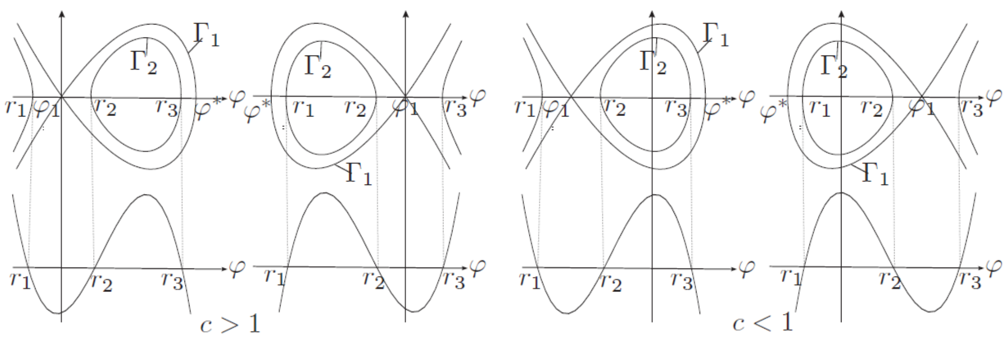

Figure 3: The phase portraits of homoclinic and periodic orbits under conditon $g = 0$

When $(b+k)c < 0$ and $c > 1$, the system has homoclinic orbits Γ_1 (see Fig.3). In $\varphi - y$ plane, the homoclinic Γ_1 can be described by the following equation

$$y^2 = \frac{2a}{3(b+k)c}\varphi^3 + \frac{c-1}{(b+k)c}\varphi^2, \ \ \varphi \in (0, \varphi^*) \ \text{ or } \ \varphi \in (\varphi^*, 0), \tag{13}$$

where $\varphi^* = -\frac{3(c-1)}{2a}$. That is

$$y = \pm\sqrt{\frac{2a}{3(b+k)c}\varphi^3 + \frac{c-1}{(b+k)c}\varphi^2}. \tag{14}$$

Substituting (14) into dφ/d$\xi = y$ and integrating along the homoclinc orbits Γ_1, we have

$$\int_{\varphi}^{\varphi^*} \frac{\mathrm{d}s}{\sqrt{\frac{2a}{3(b+k)c}s^3 - \frac{c-1}{(b+k)c}s^2}} = \pm|\xi|. \tag{15}$$

Completing the above integration yields a solution $u_1(x, y, t)$ of (5)

$$u_1(x, y, t) = \frac{-3(c-1)}{a[1 + \cosh(\sqrt{-\frac{c-1}{(b+k)c}}(x + y - ct))]}. \tag{16}$$

When $(b + k)c < 0$ and $c < 1$, the homoclinic Γ_1 can be described by the following equation

$$y^2 = \frac{2a}{3(b+k)c}\varphi^3 + \frac{c-1}{(b+k)c}\varphi^2 + h_1, \;\; \varphi \in (\varphi_1, \varphi^*) \;\text{ or }\; \varphi \in (\varphi^*, \varphi_1), \tag{17}$$

where $h_1 = -\frac{2a}{3(b+k)c}\varphi_1^3 - \frac{c-1}{(b+k)c}\varphi_1^2$, $\varphi_1 = \frac{1-c}{a}$ and $\varphi^* = -\frac{3(c-1)}{2a}$. Equation (17) can be rewritten as

$$y = \pm\sqrt{\frac{2a}{3(b+k)c}(\varphi - \varphi_1)^2(\varphi - \varphi^*)}. \tag{18}$$

Substituting (18) into dφ/d$\xi = y$ and integrating along the homoclinc orbits Γ_1, we get a solution $u_2(x, y, t)$ of equation (5) as follows

$$u_2(x, y, t) = \frac{(1-c)[-2 + \cosh(\sqrt{\frac{1-c}{(b+k)c}}(x + y - ct))]}{a[1 + \cosh(\sqrt{\frac{1-c}{(b+k)c}}(x + y - ct))]}. \tag{19}$$

Remark. The solutions u_1 and u_2 are bright soliton solutions when $a < 0$, and dark soliton solutions when $a > 0$. When $(b + k)c > 0$, there are also homoclinic orbits (see Fig.2). It is same to the above solving process that we can get exact expressions of solitary wave solutions according to homoclinic orbits as u_1 and u_2.

3.2 The case integral constant $g \neq 0$

The system (9) possesses homoclinic orbits (see Fig.1 and Fig.2). For simplicity we solve solitary wave solutions under $(b + k)c < 0$. These homoclinic orbits can be expressed by

$$y^2 = \frac{2a}{3(b+k)c}\varphi^3 - \frac{1-c}{(b+k)c}\varphi^2 + \frac{2g}{(b+k)c}\varphi + 2h_1, \tag{20}$$

where $h_1 = H(\varphi_1, 0)$ denotes Hamiltonian at point $(\varphi_1, 0)$ according to equation (10). When $(b + k)c/a < 0$, the homoclinic orbits have a double zero point φ_1 and a zero point φ_3 on $\varphi-$axis(see Fig.4), so (20) can be rewritten as

$$y^2 = \frac{2a}{3(b+k)c}(\varphi - \varphi_1)^2(\varphi - \varphi_3), \tag{21}$$

that is

$$y = \pm\sqrt{\frac{2a}{3(b+k)c}(\varphi - \varphi_1)^2(\varphi - \varphi_3)}. \tag{22}$$

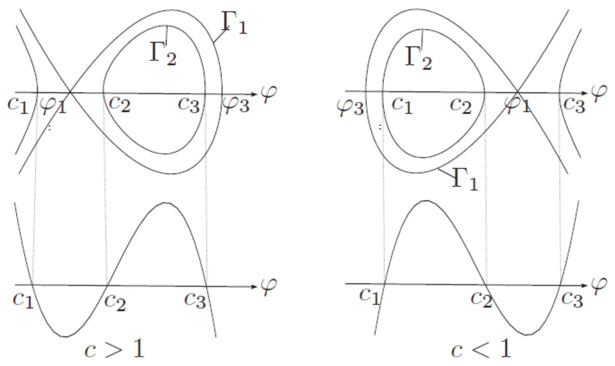

Figure 4: The phase portraits of homoclinic and periodic orbits under conditon $g \neq 0$

Substituting (22) into dφ/d$\xi = y$ and integrating along homoclinic orbits, we get

$$\int_{\varphi}^{\varphi_3} \frac{\mathrm{d}s}{\sqrt{\frac{2a}{3(b+k)c}(s-\varphi_1)^2(s-\varphi_3)}} = |\xi|, \tag{23}$$

where $\varphi_1 = \frac{1-c+\sqrt{\Delta}}{2a}$ and $\varphi_3 = \frac{1-c-2\sqrt{\Delta}}{2a}$, then completing (23) we get the following solution

$$u_3(x,y,t) = \frac{(1-c+\sqrt{\Delta})\cosh\sqrt{\frac{\sqrt{\Delta}}{(b+k)c}}(x+y-ct)+1-c-5\sqrt{\Delta}}{2a(\cosh\sqrt{\frac{\sqrt{\Delta}}{(b+k)c}}(x+y-ct)+1)}. \tag{24}$$

When $(b+k)c > 0$, similarly the expressions of solitary wave solutions can be obtained as

$$u_4(x,y,t) = \frac{(1-c-\sqrt{\Delta})\cosh\sqrt{-\frac{\sqrt{\Delta}}{(b+k)c}}(x+y-ct)+1-c+5\sqrt{\Delta}}{2a(\cosh\sqrt{-\frac{\sqrt{\Delta}}{(b+k)c}}(x+y-ct)+1)}. \tag{25}$$

4 Periodic Wave Solutions in Forms of Jacobian Elliptic Function

In order to explain our work conveniently, the Jacobian elliptic function sn(k,m) with modulus m will be expressed by sn$k$ in this section . We solve the periodic wave solutions under conditions $g = 0$ and $g \neq 0$ respectively.

4.1 The case integral constant $g = 0$

If $g = 0$, then system according to equation (5) is namely (11), and it has periodic orbits (see Fig.1 and Fig.2). For simplicity, we discuss periodic waves under $(b+k)c < 0$, and the $(b+k)c > 0$ is the same. When $(b+k)c < 0$ and $c > 1$, these periodic orbits Γ_2(see Fig.3) satisfy (12), where $h_1 < h < h_2$(or $h_2 < h < h_1$). Let

$$f_1(\varphi) = \frac{2a}{3(b+k)c}\varphi^3 - \frac{1-c}{(b+k)c}\varphi^2 + 2h, \tag{26}$$

when $h_1 < h < h_2$(or $h_2 < h < h_1$), by ShenJin Theorem [14] we can distinguish that f_1 has three different real points. Let $r_1 < r_2 < r_3$ are three different real zero points of $f_1(\varphi)$, then equation (12) can be rewritten as

$$y^2 = \frac{2a}{3(b+k)c}(\varphi - r_1)(\varphi - r_2)(\varphi - r_3) \tag{27}$$

where $r_1 < 0 < r_2 < \varphi < r_3$ when $a < 0$, and $r_1 < \varphi < r_2 < 0 < r_3$ when $a > 0$(see Fig.3). The expressions of the periodic orbits Γ_2 are given by

$$y = \pm\sqrt{\frac{2a}{3(b+k)c}(\varphi - r_1)(\varphi - r_2)(\varphi - r_3)}, (r_1 < r_2 \leq \varphi \leq r_3), \tag{28}$$

or

$$y = \pm\sqrt{\frac{2a}{3(b+k)c}(\varphi - r_1)(\varphi - r_2)(\varphi - r_3)}, (r_1 \leq \varphi \leq r_2 < r_3), \tag{29}$$

respectively. Substituting (28) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating along periodic orbit Γ_2, we get

$$\int_{\varphi}^{r_3} \frac{\mathrm{d}s}{\sqrt{(r_3 - s)(s - r_1)(s - r_2)}} = \sqrt{-\frac{2a}{3(b+k)c}}|\xi|, (r_1 < r_2 \leq \varphi < r_3). \tag{30}$$

By formula 236 in [15],we have

$$g_1 \mathrm{sn}^{-1}(\sin\psi_1, m_5) = \sqrt{-\frac{2a}{3(b+k)c}}|\xi|, \tag{31}$$

where $g_1 = \frac{2}{\sqrt{r_3 - r_1}}$, $\sin\psi_1 = \sqrt{\frac{r_3 - \varphi}{r_3 - r_2}}$ and $m_5 = \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}$. Solving (31) we get

$$\varphi = r_3 - (r_3 - r_2)\mathrm{sn}^2\sqrt{-\frac{a(r_3 - r_1)}{6(b+k)c}}\xi, \tag{32}$$

That is

$$u_5(x, y, t) = r_3 - (r_3 - r_2)\mathrm{sn}^2\sqrt{-\frac{a(r_3 - r_1)}{6(b+k)c}}(x + y - ct), \tag{33}$$

where the modulus of sn is $m_5 = \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}$. Similarly, Substituting (29) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating along orbits Γ_2, we have

$$u_6(x, y, t) = r_1 + (r_2 - r_1)\mathrm{sn}^2\sqrt{\frac{a(r_3 - r_1)}{6(b+k)c}}(x + y - ct), \tag{34}$$

where the modulus of sn is $m_6 = \sqrt{\frac{r_2 - r_1}{r_3 - r_1}}$. When$(b + k)c < 0$ and $c < 1$, periodic wave solutions can be obtained as u_5 and u_6 as the above procedure.

4.2 The case integral constant $g \neq 0$

If $g \neq 0$, then system according to equation (5) is namely (11), and it has periodic orbits (see Fig.1 and Fig.2). Their expressions are (10) on the $\varphi - y$ plane, where $h_1 < h < h_2$(or $h_1 < h < h_2$). Let

$$f_2(\varphi) = \frac{2a}{3(b+k)c}\varphi^3 - \frac{1-c}{(b+k)c}\varphi^2 + \frac{2g}{(b+k)c}\varphi + 2h.$$

If $g \neq 0$ and $h_1 < h < h_2$(or $h_2 < h < h_1$), then the function $f_2(\varphi)$ must have three different real zero points. In fact, under above conditions,

$h_1 = H(\varphi_1, 0) = -\frac{a}{3(b+k)c}\varphi_1^3 + \frac{1-c}{2(b+k)c}\varphi_1^2 - \frac{g}{(b+k)c}\varphi_1 = -\frac{1}{2}f_2(\varphi_1) + h,$

$h_2 = H(\varphi_2, 0) = -\frac{a}{3(b+k)c}\varphi_2^3 + \frac{1-c}{2(b+k)c}\varphi_2^2 - \frac{g}{(b+k)c}\varphi_2 = -\frac{1}{2}f_2(\varphi_2) + h,$

So $f_2(\varphi_1) \cdot f_2(\varphi_2) = 4(h - h_1)(h - h_2) < 0$. For $f_2(\varphi)$, we have $f_2(-\infty) > 0$,$f_2(\varphi_1) < 0$,$f_2(\varphi_2) > 0$ and $f_2(+\infty) < 0$. Again,$f_2'(\varphi) = \frac{2a}{(b+k)c}(\varphi - \varphi_1)(\varphi - \varphi_2)$, which is monotonous in the intervals $(-\infty, \varphi_1)$,(φ_1, φ_2) and $(\varphi_2, +\infty)$. By zero point theorem of continuous function, there must be one real zero point of $f_2(\varphi)$ lies in each of the three intervals. Let $c_1 < c_2 < c_3$ are three different real zero points of $f_2(\varphi)$. Then (10) can be rewritten as

$$y^2 = \frac{2a}{3(b+k)c}(\varphi - c_1)(\varphi - c_2)(\varphi - c_3) \tag{35}$$

where $c_1 < \varphi_1 < c_2 < \varphi_2 < c_3$. When $(b+k)c < 0$ and $a < 0$, the expression of periodic orbits are

$$y = \pm\sqrt{\frac{2a}{3(b+k)c}(\varphi - c_1)(\varphi - c_2)(\varphi - c_3)}, (c_1 < c_2 \leq \varphi \leq c_3). \tag{36}$$

When $(b+k)c < 0$ and $a > 0$, the expression of periodic orbits are

$$y = \pm\sqrt{\frac{2a}{3(b+k)c}(\varphi - c_1)(\varphi - c_2)(\varphi - c_3)}, (c_1 \leq \varphi \leq c_2 < c_3). \tag{37}$$

Substituting (36) and (37) into $\frac{d\varphi}{d\xi} = y$ and integrating along periodic orbits respectively, it is same to the proceeding for solving u_5 and u_6, we can get the corresponding periodic solutions as follows

$$u_7(x, y, t) = c_3 - (c_3 - c_2)\text{sn}^2\sqrt{-\frac{a(c_3 - c_1)}{6(b+k)c}}(x + y - ct), \tag{38}$$

and

$$u_8(x, y, t) = c_1 + (c_2 - c_1)\text{sn}^2\sqrt{\frac{a(c_3 - c_1)}{6(b+k)c}}(x + y - ct), \tag{39}$$

where the modulus for sn are $m_7 = \sqrt{\frac{c_3-c_2}{c_3-c_1}}$ in $(38)$and $m_8 = \sqrt{\frac{c_2-c_1}{c_3-c_1}}$ in (39).

Compared with the solutions (30) and (31) in [16], and periodic wave solutions in [17], we find that the periodic wave solutions are new.

5 Relationship Between Solitary Waves and Periodic Waves

In Sec.3 and Sec.4, we obtain the solitary wave and periodic wave solutions. With further study, we find that there exists a colse relationship between these two kind of solutions, that is, the solitary wave solutions are limits of the periodic ones in the sense of modulus of Jacobian elliptic functions approach 1. The results are detailed as follows.

Theroem. Let $u_i(i = 1, 2, \cdots, 8)$ are solutions of equation (5), $, a, b, c, k$ and g are parameters in (8), and $m_i(i = 5, 6, 7, 8)$ are modulus of Jacobian elliptic function sn, then we have the following conclusions:

(1). When $g = 0$ and $\frac{1-c}{(b+k)c} < 0$, for modulus $m_i \to 1(i = 5, 6)$, the periodic wave solutions u_5 and u_6 degenerate solitary wave solution u_1;

(2). When $g = 0$ and $\frac{1-c}{(b+k)c} > 0$, for modulus $m_i \to 1(i = 5, 6)$, the periodic wave solutions u_5 and

u_6 degenerate solitary wave solution u_2;
(3). When $g \neq 0$ and $(b+k)c/a < 0$, for modulus $m_7 \to 1$, the periodic wave solution u_7 degenerates solitary wave solution u_4;
(4). When $g \neq 0$ and $(b+k)c/a > 0$, for modulus $m_8 \to 1$, the periodic wave solution u_8 degenerates solitary wave solution u_3.

For the sake of simplicity, here we only prove (1) and (3) , the rest cases are the same. In the following proofs, we use the property of elliptic function that sn $\to \tanh$ when the modulus $m \to 1$ [17],[18].

Proof of (1). When $m_5 = \sqrt{\frac{r_3-r_2}{r_3-r_1}} \to 1$, it means $r_1 = r_2$ and sn $= \tanh$, then we calculate

$$r_1 = r_2 = 0 \quad \text{and} \quad r_3 = \frac{3(1-c)}{2a},$$

substituting $r_i (i = 1, 2, 3)$ into u_5 admits to u_1 as follows

$$\begin{aligned}
u_5(x,y,t) &= r_3 - (r_3 - r_2)\text{sn}^2\sqrt{-\frac{a(r_3-r_1)}{6(b+k)c}}(x+y-ct) \\
&= \frac{3(1-c)}{2a} - \frac{3(1-c)}{2a}\tanh^2\sqrt{-\frac{(1-c)}{4(b+k)c}}(x+y-ct) \\
&= \frac{3(1-c)}{2a}[1-\tanh^2\sqrt{-\frac{(1-c)}{4(b+k)c}}(x+y-ct)] \\
&= \frac{3(1-c)}{2a}\frac{1}{\cosh^2\sqrt{-\frac{(1-c)}{4(b+k)c}}(x+y-ct)} \\
&= \frac{3(1-c)}{2a}\frac{1}{\frac{1}{2}[\cosh\sqrt{-\frac{(1-c)}{(b+k)c}}(x+y-ct)+1]} \\
&= \frac{3(1-c)}{a[1+\cosh\sqrt{-\frac{(1-c)}{(b+k)c}}(x+y-ct)]} \\
&= u_1(x,y,t).
\end{aligned}$$

When $m_6 = \sqrt{\frac{r_2-r_1}{r_3-r_1}} \to 1$, it means $r_2 = r_3$, then we calculate $r_2 = r_3 = 0$ and $r_1 = \frac{3(1-c)}{2a}$, substituting $r_i(i = 1, 2, 3)$ into u_6 we get $u_6 = u_1$.

Proof of (3). When $m_7 = \sqrt{\frac{c_3-c_2}{c_3-c_1}} \to 1$, it means $c_1 = c_2$ and sn $= \tanh$, then we calculate

$c_1 = c_2 = \frac{1-c-\sqrt{\Delta}}{2a}$ and $c_3 = \frac{1-c+2\sqrt{\Delta}}{2a}$, substituting $c_i(i=1,2,3)$ into u_7 admits to u_4 as follows

$$\begin{aligned} u_7(x,y,t) &= c_3 - (c_3-c_2)\mathrm{sn}^2\sqrt{-\frac{a(c_3-c_1)}{6(b+k)c}}(x+y-ct) \\ &= \frac{1-c+2\sqrt{\Delta}}{2a} - \frac{3\sqrt{\Delta}}{2a}\tanh^2\sqrt{\frac{-\sqrt{\Delta}}{4(b+k)c}}(x+y-ct) \\ &= \frac{1-c-\sqrt{\Delta}+3\sqrt{\Delta}(1-\tanh^2\sqrt{\frac{-\sqrt{\Delta}}{4(b+k)c}}(x+y-ct))}{2a} \\ &= \frac{1-c-\sqrt{\Delta}}{2a} + \frac{3\sqrt{\Delta}}{a[1+\cosh\sqrt{\frac{-\sqrt{\Delta}}{(b+k)c}}(x+y-ct))]} \\ &= \frac{(1-c-\sqrt{\Delta})\cosh\sqrt{-\frac{\sqrt{\Delta}}{(b+k)c}}(x+y-ct)+1-c+5\sqrt{\Delta}}{2a[\cosh\sqrt{-\frac{\sqrt{\Delta}}{(b+k)c}}(x+y-ct)+1]} \\ &= u_4(x,y,t). \end{aligned}$$

6 Conclusion

The results in this paper means that the bifurcation method of dynamical system is effective for solving nonlinear evolution equations, and it can be widely used to other nonlinear equations. Besides solitary and periodic waves, the method can be used to seek other kind waves such as kink waves, peakons, compactons, cuspons and so on. We will also study other kind solutions in the future.

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Competing Interests

The authors declare that no competing interests exist.

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