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Hyperstability of a Cauchy-Jensen Type Functional Equation

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Abstract

In this paper, we establish some hyperstability results concerning the Cauchy - Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x)$$

in Banach spaces.

Keywords: Hyperstability; Cauchy-Jensen equation; fixed point theorem

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1 Introduction

In 1940, Ulam [1] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?
Let X and Y be Banach spaces with norms $\|.\|$ and $\|.\|$, respectively. In 1941, Hyers [2] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon,$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x)-T(x)\| \leq \epsilon,$$

for all $x \in X$. In 1978, Rassias [3] introduced the following inequality, that we call Cauchy-Rassias inequality. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \theta(\|x\|^p+\|y\|^p),$$

for all $x, y \in X$. Rassias [3] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x)-T(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p,$$

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for all $x \in X$. The above inequality has produced a lot of influence on the development of what we now call the *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians (see [4], [5], [6], [7], [8] and [9]).

Recently, interesting results concerning Cauchy-Jensen functional equation

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x) \tag{1}$$

have been obtained in [10], [11], [12], [13] and [14].

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. It seems that the first hyperstability result was published in [15] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [16]. Quite often the hyperstability is confused with superstability, which admits also bounded functions.

Numerous papers on this subject have been published and we refer to [17], [18], [19] and [20].

Throughout this paper, we present the hyperstability results for the Cauchy-Jensen functional equation (1) in Banach spaces.

The method of the proofs used in the main results is based on a fixed point result that can be derived from [Theorem 1 [21]]. To present it we need the following three hypothesis:

(**H1**) X is a nonempty set, Y is a Banach space, $f_1, ..., f_k : X \longrightarrow X$ and $L_1, ..., L_k : X \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{k} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \qquad \xi, \mu \in Y^X, \quad x \in X.$$

(**H3**) $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in \mathbb{R}_+^X, \quad x \in X.$$

The following theorem is the basic tool in this paper. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

Theorem 1. *Let hypotheses (**H1**)-(**H3**) be valid and functions $\varepsilon : X \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \le \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in X.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x), \qquad x \in X.$$

Moreover

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \qquad x \in X.$$

2 Hyperstability Results

The following theorems are the main results in this paper and concern the hyperstability of equation (1).

Theorem 2. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p, q \in \mathbb{R}$, $p+q<0$ and let $f : X \longrightarrow Y$ satisfy*

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right\| \leq c\|x\|^p \cdot \|y\|^q \tag{2}$$

for all $x, y \in X \setminus \{0\}$. Then f is Cauchy-Jensen on $X \setminus \{0\}$.

Proof. Since $p+q<0$, one of p, q must be negative. Assume that $q<0$ and replace y by mx, where $m \in \mathbb{N}$, in (2). We get that

$$\left\| f\left(\frac{1+m}{2}x\right) + f\left(\frac{1-m}{2}x\right) - f(x) \right\| \leq cm^q\|x\|^{p+q} \tag{3}$$

for all $x \in X \setminus \{0\}$. Write

$$\mathcal{T}_m\xi(x) := \xi\left(\frac{1+m}{2}x\right) + \xi\left(\frac{1-m}{2}x\right), \qquad x \in X \setminus \{0\}, \xi \in Y^{X\setminus\{0\}},$$

$$\varepsilon_m(x) := cm^q\|x\|^{p+q}, \qquad x \in X \setminus \{0\},$$

then (3) takes the following form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

Define

$$\Lambda_m\eta(x) := \eta\left(\frac{1+m}{2}x\right) + \eta\left(\frac{1-m}{2}x\right), \qquad x \in X \setminus \{0\}, \eta \in \mathbb{R}_+^{X\setminus\{0\}}.$$

Then it is easily seen that Λ_m has the form described in (**H3**) with $k=2$ and $f_1(x) = \frac{1+m}{2}x$, $f_2(x) = \frac{1-m}{2}x$, $L_1(x) = L_2(x) = 1$ for $x \in X \setminus \{0\}$.
Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$ and $x \in X \setminus \{0\}$, we get that

$$\begin{aligned}\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| &= \left\| \xi\left(\frac{1+m}{2}x\right) + \xi\left(\frac{1-m}{2}x\right) - \mu\left(\frac{1+m}{2}x\right) - \mu\left(\frac{1-m}{2}x\right) \right\| \\ &\leq \left\| (\xi-\mu)\left(\frac{1+m}{2}x\right) \right\| + \left\| (\xi-\mu)\left(\frac{1-m}{2}x\right) \right\| = \sum_{i=1}^{2} L_i(x)\|(\xi-\mu)(f_i(x))\|.\end{aligned}$$

So, (**H2**) is valid. Next, we can find $m_0 \in \mathbb{N}$ such that

$$\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q} < 1 \qquad \text{for all} \quad m \geq m_0.$$

Therefore, we obtain that

$$\begin{aligned}\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\ &= cm^q\|x\|^{p+q} \sum_{n=0}^{\infty} \left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q} \right)^n \\ &= \frac{cm^q\|x\|^{p+q}}{1 - \left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q}}, \qquad x \in X \setminus \{0\}, m \geq m_0.\end{aligned}$$

Hence, according to Theorem 1, for each $m \geq m_0$ there exists a unique solution $A_m : X \setminus \{0\} \to Y$ of the equation

$$A_m(x) = A_m\left(\frac{1+m}{2}x\right) + A_m\left(\frac{1-m}{2}x\right), \qquad x \in X \setminus \{0\}$$

such that

$$\|f(x) - A_m(x)\| \leq \frac{cm^q\|x\|^{p+q}}{1 - \left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q}}, \qquad x \in X \setminus \{0\}, m \geq m_0.$$

Moreover,

$$A_m(x) := \lim_{n\to\infty} \mathcal{T}_m^n f(x), \qquad x \in X \setminus \{0\}.$$

To prove that A_m satisfies the Cauchy-Jensen equation on $X \setminus \{0\}$, we show that

$$\left\|\mathcal{T}_m^n f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^n f(x)\right\| \leq c\left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q}\right)^n \|x\|^p\|y\|^q \qquad (4)$$

for every $x, y \in X \setminus \{0\}$ and every $n \in \mathbb{N}_0$.

If $n = 0$, then (4) is simply (2). So, take $r \in \mathbb{N}_0$ and suppose that (4) holds for $n = r$. Then

$$\left\|\mathcal{T}_m^{r+1} f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^{r+1} f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^{r+1} f(x)\right\| = \left\|\mathcal{T}_m^r f\left(\frac{1+m}{2}\frac{x+y}{2}\right) + \mathcal{T}_m^r f\left(\frac{1-m}{2}\frac{x+y}{2}\right)\right.$$

$$\left. + \mathcal{T}_m^r f\left(\frac{1+m}{2}\frac{x-y}{2}\right) + \mathcal{T}_m^r f\left(\frac{1-m}{2}\frac{x-y}{2}\right) - \mathcal{T}_m^r f\left(\frac{1+m}{2}x\right) - \mathcal{T}_m^r f\left(\frac{1-m}{2}x\right) \right\|$$

$$\leq \left\|\mathcal{T}_m^n f\left(\frac{1+m}{2}\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{1+m}{2}\frac{x-y}{2}\right) - \mathcal{T}_m^n f(\frac{1+m}{2}x)\right\|$$
$$+ \left\|\mathcal{T}_m^n f\left(\frac{1-m}{2}\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{1-m}{2}\frac{x-y}{2}\right) - \mathcal{T}_m^n f(\frac{1-m}{2}x)\right\|$$

$$\leq c\left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q}\right)^r \left(\left\|\frac{1+m}{2}x\right\|^p \cdot \left\|\frac{1+m}{2}y\right\|^q + \left\|\frac{1-m}{2}x\right\|^p \cdot \left\|\frac{1-m}{2}y\right\|^q\right)$$

$$= c\left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q}\right)^{r+1} \|x\|^p \cdot \|y\|^q, \qquad x, y \in X \setminus \{0\}.$$

Thus, by induction we show that (4) holds for all $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (4), we obtain that

$$A_m(x) = A_m\left(\frac{x+y}{2}x\right) + A_m\left(\frac{x-y}{2}x\right), \qquad x, y \in X \setminus \{0\}.$$

So, we obtain a sequence $\{A_m\}_{m\geq m_0}$ of Cauchy-Jensen functions on $X \setminus \{0\}$ such that

$$\|f(x) - A_m(x)\| \leq \frac{cm^q\|x\|^{p+q}}{1 - \left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q}}, \qquad x \in X \setminus \{0\}.$$

It follows, with $m \longrightarrow \infty$, that f is Cauchy-Jensen on $X \setminus \{0\}$.

$\square$

In a similar way we can prove the following two theorems.

Theorem 3. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p, q \in \mathbb{R}$, $p+q > 0$ and let $f : X \longrightarrow Y$ satisfies (2). Then f is Cauchy-Jensen on $X \setminus \{0\}$.*

Proof. Replacing y by $\frac{1}{m}x$ in (2), where $m \in \mathbb{N}$, we get

$$\left\| f\left(\frac{m+1}{2m}x\right) + f\left(\frac{m-1}{2m}x\right) - f(x) \right\| \leq \frac{c}{m^q}\|x\|^{p+q} \tag{5}$$

for all $x \in X \setminus \{0\}$. Define operators $\mathcal{T}_m : Y^{X\setminus\{0\}} \to Y^{X\setminus\{0\}}$ and $\Lambda_m : \mathbb{R}_+^{X\setminus\{0\}} \to \mathbb{R}_+^{X\setminus\{0\}}$ by

$$\mathcal{T}_m\xi(x) := \xi\left(\frac{m+1}{2m}x\right) + \xi\left(\frac{m-1}{2m}x\right), \qquad x \in X \setminus \{0\}, \quad \xi \in Y^{X\setminus\{0\}},$$

$$\Lambda_m\delta(x) := \delta\left(\frac{m+1}{2m}x\right) + \delta\left(\frac{m-1}{2m}x\right), \qquad x \in X \setminus \{0\}, \quad \delta \in \mathbb{R}_+^{X\setminus\{0\}}.$$

Then it is easily seen that Λ_m has the form described in (**H3**) with $k = 2$ and

$$f_1(x) = \frac{m+1}{2m}x, \quad f_2(x) = \frac{m-1}{2m}x, \quad L_1(x) = L_2(x) = 1$$

for $x \in X \setminus \{0\}$. Further, (5) can be written in the form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in X \setminus \{0\},$$

with

$$\varepsilon_m(x) := \frac{c}{m^q}\|x\|^{p+q}.$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$ and $x \in X \setminus \{0\}$, we have

$$\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| = \left\|\xi\left(\frac{m+1}{2m}x\right) + \xi\left(\frac{m-1}{2m}x\right) - \mu\left(\frac{m+1}{2m}x\right) - \mu\left(\frac{m-1}{2m}x\right)\right\|$$

$$\leq \left\|(\xi-\mu)\left(\frac{m+1}{2m}x\right)\right\| + \left\|(\xi-\mu)\left(\frac{m-1}{2m}x\right)\right\| = \sum_{i=1}^{2} L_i(x)\|(\xi-\mu)(f_i(x))\|$$

and hypothesis (**H2**) holds, too. We can find $m_0 \in \mathbb{N}$ such that

$$\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q} < 1 \quad \text{for all} \quad m \geq m_0.$$

Note yet that we have

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= \frac{c}{m^q}\|x\|^{p+q} \sum_{n=0}^{\infty} \left(\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q} \right)^n \\
&= \frac{c\|x\|^{p+q}}{m^q(1 - \left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q})}, \qquad x \in X \setminus \{0\}, m \geq m_0.
\end{aligned}$$

Consequently, in view of Theorem 1, for each $m \geq m_0$ there exists a unique solution $A_m : X \setminus \{0\} \to Y$ of the equation

$$A_m(x) = A_m\left(\frac{m+1}{2m}x\right) + A_m\left(\frac{m-1}{2m}x\right), \qquad x \in X \setminus \{0\}$$

such that

$$\|f(x) - A_m(x)\| \le \frac{c\|x\|^{p+q}}{m^q(1 - \left|\frac{m+1}{2m}\right|^{p+q} - \left|\frac{m-1}{2m}\right|^{p+q})}, \qquad x \in X \setminus \{0\}, m \ge m_0.$$

Moreover,

$$A_m(x) := \lim_{n\to\infty} \mathcal{T}_m^n f(x), \qquad x \in X \setminus \{0\}.$$

Then we show

$$\left\|\mathcal{T}_m^n f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^n f(x)\right\| \le c\left(\left|\frac{m+1}{2m}\right|^{p+q} - \left|\frac{m-1}{2m}\right|^{p+q}\right)^n \|x\|^p\|y\|^q \tag{6}$$

for every $x, y \in X \setminus \{0\}$ and every $n \in \mathbb{N}_0$.

If $n = 0$, then (6) is simply (2). So, take $r \in \mathbb{N}_0$ and suppose that (6) holds for $n = r$. Then

$$\begin{aligned}
\left\|\mathcal{T}_m^{r+1} f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^{r+1} f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^{r+1} f(x)\right\| &= \left\|\mathcal{T}_m^r f\left(\frac{m+1}{2m}\frac{x+y}{2}\right) + \mathcal{T}_m^r f\left(\frac{m-1}{2m}\frac{x+y}{2}\right)\right. \\
&+\mathcal{T}_m^r f\left(\frac{m+1}{2m}\frac{x-y}{2}\right) + \mathcal{T}_m^r f\left(\frac{m-1}{2m}\frac{x-y}{2}\right) - \mathcal{T}_m^r f\left(\frac{m+1}{2m}x\right) - \mathcal{T}_m^r f\left(\frac{m-1}{2m}x\right)\| \\
&\le \left\|\mathcal{T}_m^n f\left(\frac{m+1}{2m}\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{m+1}{2m}\frac{x-y}{2}\right) - \mathcal{T}_m^n f(\frac{m+1}{2m}x)\right\| \\
&+ \left\|\mathcal{T}_m^n f\left(\frac{m-1}{2m}\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{m-1}{2m}\frac{x-y}{2}\right) - \mathcal{T}_m^n f(\frac{m-1}{2m}x)\right\| \\
&\le c\left(\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q}\right)^r \left(\left\|\frac{m+1}{2m}x\right\|^p \cdot \left\|\frac{m+1}{2m}y\right\|^q + \left\|\frac{m-1}{2m}x\right\|^p \cdot \left\|\frac{m-1}{2m}y\right\|^q\right) \\
&= c\left(\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q}\right)^{r+1} \|x\|^p \cdot \|y\|^q, \qquad x, y \in X \setminus \{0\}.
\end{aligned}$$

Thus, by induction we show that (6) holds for all $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (6), we obtain that

$$A_m(x) = A_m\left(\frac{x+y}{2}x\right) + A_m\left(\frac{x-y}{2}x\right), \qquad x, y \in X \setminus \{0\}.$$

So, we obtain a sequence $\{A_m\}_{m \ge m_0}$ of Cauchy-Jensen functions on $X \setminus \{0\}$ such that

$$\|f(x) - A_m(x)\| \le \frac{c\|x\|^{p+q}}{m^q(1 - \left|\frac{m+1}{2m}\right|^{p+q} - \left|\frac{m-1}{2m}\right|^{p+q})}, \qquad x \in X \setminus \{0\}.$$

It follows, with $m \longrightarrow \infty$, that f is Cauchy-Jensen on $X \setminus \{0\}$. □

Theorem 4. *Let X be a normed space, Y be a Banach space, $c \ge 0$, $p < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\left\|f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x)\right\| \le c(\|x\|^p + \|y\|^p) \tag{7}$$

for all $x, y \in X \setminus \{0\}$. Then f is Cauchy-Jensen on $X \setminus \{0\}$.

Proof. Replacing x by $(m+2)x$ and y by $-mx$, where $m \in \mathbb{N}$, in (7), we obtain

$$\|f(x)+f((m+1)x)-f((m+2)x)\| \le c((m+2)^p+m^p)\|x\|^p \tag{8}$$

for all $x \in X \setminus \{0\}$. Write

$$\mathcal{T}_m\xi(x) := \xi((m+2)x)-\xi((m+1)x), \qquad x \in X \setminus \{0\}, \xi \in Y^{X\setminus\{0\}},$$

$$\varepsilon_m(x) := c((m+2)^p+m^p)\|x\|^p.$$

Inequality (8) takes the following form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following linear operator $\Lambda_m : \mathbb{R}_+^{X\setminus\{0\}} \longrightarrow \mathbb{R}_+^{X\setminus\{0\}}$ which is defined by

$$\Lambda_m\eta(x) := \eta((m+2)x)+\eta((m+1)x), \qquad \eta \in \mathbb{R}_+^{X\setminus\{0\}}, x \in X \setminus \{0\}$$

has the form described in (**H3**) with $k=2$ and $f_1(x)=(m+2)x, f_2(x)=(m+1)x, L_1(x)=L_2(x)=1$, for $x \in X \setminus \{0\}$.

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$

$$\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x)\| = \|\xi\left((m+2)x\right)-\xi\left((m+1)x\right)-\mu\left((m+2)x\right)+\mu\left((m+1)x\right)\|$$

$$\le \|(\xi-\mu)\left((m+2)x\right)\|+\|(\xi-\mu)\left((m+1)x\right)\| = \sum_{i=1}^{2} L_i(x)\|(\xi-\mu)(f_i(x))\|.$$

So, (**H2**) is valid. Now, we can find $m_0 \in \mathbb{N}$ such that

$$(m+2)^p+(m+1)^p < 1 \qquad \text{for all} \quad m \ge m_0.$$

Therefore, we obtain that

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= c((m+2)^p+m^p)\sum_{n=0}^{\infty} \Lambda_m^n\left(\|(m+2)x\|^p+\|(m+1)x\|^p\right) \\
&= c((m+2)^p+m^p)\|x\|^p \sum_{n=0}^{\infty}\left((m+2)^p+(m+1)^p\right)^n \\
&= \frac{c((m+2)^p+m^p)\|x\|^p}{1-(m+2)^p-(m+1)^p}, \qquad x \in X \setminus \{0\}, m \ge m_0.
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.

□

3 Conclusion

This paper indeed presents a relationship between three various disciplines: the theory of Banach spaces, the theory of stability of functional equations and the fixed point theory. We established some hyperstability results concerning a Cauchy-Jensen functional equation in Banach spaces by using fixed point theorem which given by Brzdęk J. Chudziak J. and Páles Zs. [21].

Competing Interests

The authors declare that no competing interests exist.

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