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Co-$*^n$-tuple of Contravariant Functors

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Abstract

In this work we generalize the concept of Co-$*^n$-modules to the concept of Co-$*^n$-tuple of Contravariant Functors between abelian categories.

Keywords: Co-$*^n$-modules; contravariant functor; right adjoint functors; duality.

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1 Introduction

For a unital associative ring A, a fixed right A-module M , and $D = End_A(M)$, let fgd-tl(M_A) denote the class of all torsionless right A-modules whose M-dual are finitely generated over D and fg-tl($_DM$) denote the class of all finitely generated torsionless left D-modules. M is called costar module if

$$Hom_A(-, M) : fgd\text{-}tl(M_A) \rightleftarrows fg\text{-}tl(_DM) : Hom_D(-, M)$$

is a duality. Costar modules were introduced by Colby and Fuller in [1]. M is said to be an r-costar module provided that any exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

such that X and Y are M-reflexive, remains exact after applying the functor $Hom_A(-, M)$ if and only if Z is M-reflexive. The notion of r-costar module was introduced by Liu and Zhang in [2]. We say that a right A-module X is n-finitely M-copresented if there exists a long exact sequence

$$0 \longrightarrow X \longrightarrow M^{k_0} \longrightarrow M^{k_1} \longrightarrow ... \longrightarrow M^{k_{n-1}}$$

such that n is a positive integer and k_i are positive integers for $0 \leq i \leq n-1$. The class of all n-finitely M-copresented modules is denoted by n-$cop(M)$. We say that a right A-module M is a finitistic n-self-cotilting module provided that n-$cop(M) = (n+1)$-$cop(M)$ and for any exact sequence

$$0 \longrightarrow X \longrightarrow M^n \longrightarrow Z \longrightarrow 0$$

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such that $Z \in n\text{-}cop(M)$ and n is a positive integer, remains exact after applying the functor $Hom_A(-, M)$. Finitistic n-self-cotilting modules were introduced by Breaz in [3]. In [4], L. Yao and J. Chen introduced the concept of co-$*^n$-modules, which is a generalization of finitistic n-self-cotilting modules to the infinite case, i.e. we say that a right A-module M is a co-$*^n$-module provided that $n\text{-}cop(M) = (n+1)\text{-}cop(M)$ and for any exact sequence

$$0 \longrightarrow X \longrightarrow M^I \longrightarrow Z \longrightarrow 0$$

such that $Z \in n\text{-}cop(M)$ and I is a set, remains exact after applying the functor $Hom_A(-, M)$.

In [5] Castaño-Iglesias generalizes the notion of costar module to Grothendieck categories. Pop in [6] generalizes the notion of finitistic n-self-cotilting module to finitistic n-F-cotilting object in abelian categories and he describes a family of dualities between some special abelian categories. Breaz and Pop in [7] generalize a duality exhibited in [3, Theorem 2.8] to abelian categories.

In [8], the author generalizes the notion of r-costar module to r-costar pair of contravariant functors between abelian categories, by generalizing the work in [2]. In this paper we generalize the work in [4] by generalizing the notion of Co-$*^n$-modules to a Co-$*^n$-tuple of contravariant functors between abelian categories. We use the same technique of proofs as in [4].

2 Preliminaries

Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be additive and contravariant functors between two abelian categories $\mathfrak{C}$ and $\mathfrak{D}$. It is said that they are adjoint on the right if there are natural isomorphisms

$$\eta_{X,Y} : Hom_{\mathfrak{C}}(X, G(Y)) \longrightarrow Hom_{\mathfrak{D}}(Y, F(X))$$

for all $X \in \mathfrak{C}$ and all $Y \in \mathfrak{D}$. Then they induce two natural transformations $\delta : 1_{\mathfrak{C}} \longrightarrow GF$ and $\delta' : 1_{\mathfrak{D}} \longrightarrow FG$ defined by $\delta_X = \eta^{-1}_{X,F(X)}(1_{F(X)})$ and $\delta'_Y = \eta_{G(Y),Y}(1_{G(Y)})$. Moreover the following identities are satisfied for each $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$.

$$F(\delta_X) \circ \delta'_{F(X)} = 1_{F(X)} \text{ and } G(\delta'_Y) \circ \delta_{G(Y)} = 1_{G(Y)}.$$

F and G are left exact, since they are adjoint on the right. The pair (F, G) is called a duality if there are functorial isomorphisms $GF \simeq 1_{\mathfrak{C}}$ and $FG \simeq 1_{\mathfrak{D}}$. An object X of $\mathfrak{C}$ (respectively Y of $\mathfrak{D}$) is called F-reflexive (respectively, G-reflexive) in case δ_X (respectively, δ'_Y) is an isomorphism. An object X of $\mathfrak{C}$ (respectively Y of $\mathfrak{D}$) is called F-torsionless (respectively, G-torsionless) in case δ_X (respectively, δ'_Y) is a monomorphism. By Ref(F) we will denote the full subcategory of all F-reflexive objects. As well by Ref(G) we will denote the full subcategory of all G-reflexive objects. It is clear that the functors F and G induce a duality between the categories Ref(F) and Ref(G).

Let U be an object in $\mathfrak{C}$. For an object X in an abelian category $\mathfrak{C}$, we say that X is U-generated if there is an exact sequence

$$U^{(I)} \longrightarrow X \longrightarrow 0,$$

where I is an index set and if there is an exact sequence

$$U^{(I_2)} \longrightarrow U^{(I_1)} \longrightarrow X \longrightarrow 0,$$

where each I_i is an index set, then X is said to be U-presented. We say that X is U-cogenerated if there is an exact sequence

$$0 \longrightarrow X \longrightarrow U^I,$$

where I is an index set and if there is an exact sequence

$$0 \longrightarrow X \longrightarrow U^{I_1} \longrightarrow U^{I_2},$$

where each I_i is an index set, then X is said to be U-copresented. We say that X is n-U-copresented if there is an exact sequence

$$0 \longrightarrow X \longrightarrow U^{I_0} \longrightarrow U^{I_1} \longrightarrow ... \longrightarrow U^{I_{n-2}} \longrightarrow U^{I_{n-1}},$$

where each I_i is an index set and n is a positive integer. We denote by $Cogen(U)$, $Copres(U)$ and n-$Copres(U)$ the classes of all U-cogenerated, U-copresented and n-U-copresented objects respectively. It is clear that $(n+1)$-$Copres(U) \subseteq n$-$Copres(U)$, for every positive integer n.

An object U in $\mathfrak{C}$ is called co-F-small if for any set I, there is a canonical isomorphism $F(U^I) \cong F(U)^{(I)}$. The object U is called n-F-quasi-injective if for any exact sequence

$$0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0,$$

where $Y \in n$-$Copres(U)$, the sequence,

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0,$$

is exact.

Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be additive and contravariant functors between two abelian categories $\mathfrak{C}$ and $\mathfrak{D}$, such that they are adjoint on the right. Let $U \in \mathfrak{C}$ such that $V = F(U)$ is a projective object in $\mathfrak{D}$. If U is F-reflexive, the tuple (F, G, V, U) is called a Co-$*^n$-tuple, where n is a positive integer, if:
(i) U is co-F-small,
(ii) $(n+1)$-F-quasi-injective,
(iii) n-$Copres(U) = (n+1)$-$Copres(U)$.

There are two examples of contravariant functors that could satisfy the conditions to be Co-$*^n$-tuple. The first is the typical example and the second was exhibited by Castano-Iglesias in [5].

Let A be unital associative ring and M a right A-module. Suppose that $D = End_A(M)$, then it is clear that M is a D-A-bimodule. The contravariant functors $\Delta = Hom_A(,M) : Mod\text{-}A \longrightarrow D\text{-}Mod$ and $\Delta' = Hom_D(,M) : D\text{-}Mod \longrightarrow Mod\text{-}A$ are right adjoint.

Let $\mathcal{G}$ be a group. For $\mathcal{G}$-graded rings $A = \oplus_{x\in\mathcal{G}} A_x$ and $B = \oplus_{x\in\mathcal{G}} \;_x B$ we will denote by Mod_{gr}-A (respectively, by B-Mod_{gr}) the category of all $\mathcal{G}$-graded unital right A-modules (respectively, left B-modules) . For $N, M \in Mod_{gr}$-A we can consider the $\mathcal{G}$-graded abelian group $HOM_A(N, M)$, whose xth homogeneous component is

$$_xHOM_A(N, M) = \{f \in Hom_A(N, M) | f(N_y) \subseteq M_{xy}, \text{ for all } y \in \mathcal{G}\}.$$

If $N, M \in B\text{-}Mod_{gr}$ we can consider the $\mathcal{G}$-graded abelian group $HOM_B(N, M)$, whose xth homogeneous component is

$$HOM_B(N, M)_x = \{f \in Hom_B(N, M) | f({}_yN) \subseteq_{yx} M, \text{ for all } y \in \mathcal{G}\}.$$

If we suppose that $B = HOM_A(M, M) = END(M_A)$, then B is a $\mathcal{G}$-graded ring and M is a graded (B, A) -bimodule. Now we have a pair of contravariant functors

$$H_A^{gr} = HOM_A(-, M) : Mod_{gr} - A \rightleftarrows B - Mod_{gr} : HOM_B(-, M) =_B H^{gr}.$$

If for any object $X \in \mathfrak{C}$ there is a projective object $P \in \mathfrak{C}$ and an epimorphism $P \longrightarrow X \longrightarrow 0$, we say that $\mathfrak{C}$ has enough projectives. From now on we suppose that both $\mathfrak{C}$ and $\mathfrak{D}$ have enough projectives. It is clear that we can construct a projective resolution for any object X. Suppose we have a projective resolution of X in $\mathfrak{D}$.

$$P: \quad ... \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

This gives rise to the chain complex

$$0 \longrightarrow G(X) \longrightarrow G(P_0) \longrightarrow G(P_1) \longrightarrow ...,$$

for which we can compute its homology at the n-th spot (the kernel of the map from $G(P_n)$ modulo the image of the map to $G(P_n)$) and denote it by $H^n\left(G(P)\right)$. We define $R^nG(X) = H^n\left(G(P)\right)$ as the n-th right derived functors of G. The n-th right derived functors of F can be defined similarly. For the functor G and for a positive integer n, we define ${}^{\perp}T_G^{i \geqslant n} = \{X \in \mathfrak{D} : R^iG(X) = 0 \text{ for every } i \geqslant n\}$.

In the same way one can define ${}^{\perp}T_F^{i \geqslant n}$.

Let

$$0 \longrightarrow Q \xrightarrow{f} U \longrightarrow V \longrightarrow 0$$

be an exact sequence in $\mathfrak{C}$. Applying the functor F we get the exact sequence

$$0 \longrightarrow F(V) \longrightarrow F(U) \xrightarrow{p} X \longrightarrow 0, \tag{2.1}$$

where $X = \mathrm{Im}(F(f))$ and $F(f) = j \circ p$ is the canonical decomposition of $F(f)$, where $j : X \longrightarrow F(Q)$ is the inclusion map. Applying the functor G to the sequence (2.1), we have the following exact sequence

$$0 \longrightarrow G(X) \xrightarrow{G(p)} GF(U) \longrightarrow GF(V).$$

Now if we put $\alpha = G(j) \circ \delta_Q$, then

$$G(p) \circ \alpha = G(p) \circ G(j) \circ \delta_Q = G(j \circ p) \circ \delta_Q = GF(f) \circ \delta_Q = \delta_U \circ f.$$

So we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \xrightarrow{f} & U & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow_{\alpha} & & \downarrow_{\delta_U} & & \downarrow_{\delta_V} & & \\ 0 & \longrightarrow & G(X) & \xrightarrow{G(p)} & GF(U) & \longrightarrow & GF(V) & & \end{array} \tag{2.2}$$

Lemma 2.1. *[5, Lemma 1.1, Lemma 1.2]Let $F:\mathfrak{C}\longrightarrow\mathfrak{D}$ and $G:\mathfrak{D}\longrightarrow\mathfrak{C}$ be a pair of contravariant functors as above and assume that U and V are objects of $\mathfrak{C}$ and $\mathfrak{D}$, respectively. Then:*
(i) $F(X)\in Copres(F(U))$ whenever X is a U-presented object of $\mathfrak{C}$,
(ii) $G(Y)\in Copres(G(V))$ whenever Y is a V-presented object of $\mathfrak{D}$.
For the next statements, we assume that U and V are generators of $\mathfrak{C}$ and $\mathfrak{D}$, respectively.
(iii) X is F-torsionless if and only if $X\in Cogen(G(V))$, for every $X\in\mathfrak{C}$,
(iv) Y is G-torsionless if and only if $Y\in Cogen(F(U))$, for every $Y\in\mathfrak{D}$.

Lemma 2.2. *Let F and G be a pair of contravariant functors as above. Let V be a projective generator in $\mathfrak{D}$ with $G(V)=U$ and let n be a positive integer. For any $Y\in\mathfrak{D}$, if $R^iG(Y)=0$ for $1\leq i\leq n$, then $G(Y)\in(n+2)$-$Copres(U)$.*

3 Co-$*^n$-tuple Pair of Contravariant Functors

Throughout this section, let $F:\mathfrak{C}\longrightarrow\mathfrak{D}$ and $G:\mathfrak{D}\longrightarrow\mathfrak{C}$ be a pair of additive and contravariant functors which are adjoint on the right, between abelian categories. As well, let U be a F-reflexive object in $\mathfrak{C}$ with $F(U)=V$ be a projective generator in $\mathfrak{D}$. Moreover, we consider a positive integer n.

Proposition 3.1. *Suppose that (F,G,V,U) is a Co-$*^n$-tuple. Then for any $X\in n$-$Copres(U)$, δ_X is an isomorphism and $R^iG(F(X))=0$, for every $i\geq 1$.*

Proof. Let $X\in n$-$Copres(U)$. It follows that $X\in(n+1)$-$Copres(U)$, by assumptions. Hence there is an exact sequence

$$0\longrightarrow X\longrightarrow U^I\longrightarrow Y\longrightarrow 0,$$

where $Y\in n$-$Copres(U)$. Since (F,G,V,U) is a Co-$*^n$-tuple we have the exact sequence

$$0\longrightarrow F(Y)\longrightarrow F(U^I)\longrightarrow F(X)\longrightarrow 0,$$

after applying the functor F. Applying the functor G to the last sequence and taking into account that V is projective, hence $V^{(I)}=F(U)^{(I)}=F(U^I)$ is also projective, we get an exact sequence

$$0\longrightarrow GF(X)\longrightarrow GF(U^I)\longrightarrow GF(Y)\longrightarrow R^1G(F(X))\longrightarrow 0,$$

and the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \longrightarrow & U^I & \longrightarrow & Y & \longrightarrow & 0 & & \\ & & \downarrow_{\delta_X} & & \downarrow_{\delta_{U^I}} & & \downarrow_{\delta_Y} & & & & \\ 0 & \longrightarrow & GF(X) & \longrightarrow & GF(U^I) & \longrightarrow & GF(Y) & \longrightarrow & R^1G(F(X)) & \longrightarrow & 0. \end{array}$$

By Lemma 2.1, δ_Y is a monomorphism. By Snake Lemma, it follows that δ_X is an isomorphism since δ_{U^I} is an isomorphism. Then δ_Y is also an isomorphism by a similar argument. Hence, $R^1G(F(X))=0$, by commutativity of the right square. Since $Y\in n$-$Copres(U)$, $R^1G(F(Y))=0$. Then we can get the assertion inductively. □

Theorem 3.1. *The following conditions are equivalent*
(1) (F,G,V,U) is a Co-$*^n$-tuple.

(2) i) U is co-F-small,

(ii) For any exact sequence $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, where $X \in n\text{-}Copres(U)$ and I is a set, it remains exact after applying the functor F if and only if $Y \in n\text{-}Copres(U)$.

Proof. (1) $\Rightarrow$ (2) Suppose that we have an exact sequence $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, where $X \in n\text{-}Copres(U)$ and I a set. Assume that $Y \in n\text{-}Copres(U)$. Since (F, G, V, U) is a Co-$*^n$-tuple, we get the exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0.$$

Conversely, assume that the sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0$$

is exact. Applying the functor G we get the following long exact sequence

$$\begin{array}{c} 0 \longrightarrow GF(X) \longrightarrow GF(U^I) \longrightarrow GF(Y) \longrightarrow R^1G(F(X)) \longrightarrow \\ R^1G(F(U^I)) \longrightarrow R^1G(F(Y)) \longrightarrow .... \end{array} \tag{3.1}$$

By Proposition 3.1, δ_X is an isomorphism and $R^iG(F(X)) = 0$ for any $i \geq 1$. Thus, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & U^I & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow\delta_X & & \downarrow\delta_{U^I} & & \downarrow\delta_Y & & \\ 0 & \longrightarrow & GF(X) & \longrightarrow & GF(U^I) & \longrightarrow & GF(Y) & \longrightarrow & 0 \end{array}$$

It is clear, by Snake Lemma, that δ_Y is an isomorphism, which means that $Y \cong GF(Y)$. From the exactness of sequence (3.1) we conclude that $R^iG(F(Y)) \cong R^iG(F(U^I)) = 0$ for any $i \geq 1$, so by Lemma 2.2, $Y \cong GF(Y) \in n\text{-}Copres(U)$. (2)$\Rightarrow$ (1) It is enough to prove $n\text{-}Copres(U) = (n+1)\text{-}Copres(U)$. If $X \in n\text{-}Copres(U)$, then $F(X)$ is V-generated over $\mathfrak{D}$, thus by [5, Lemma 2.2], there exists an exact sequence $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, which remains exact after applying the functor F. Then $Y \in n\text{-}Copres(U)$, hence $X \in (n+1)\text{-}Copres(U)$. □

Proposition 3.2. *Suppose that (F, G, V, U) be a Co-$*^n$-tuple. Then G is an exact functor in $F(n\text{-}Copres(U))$. Moreover $F(n\text{-}Copres(U)) =^\perp T_G^{i\geqslant 1}$.*

Proof. By Proposition 3.1 we have $F(n\text{-}Copres(U)) \subseteq^\perp T_G^{i\geqslant 1}$ and G is an exact functor in $F(n\text{-}Copres(U))$. Conversely, for any $X \in^\perp T_G^{i\geqslant 1}$, by Lemma 2.2, $G(X) \in n\text{-}Copres(U)$. Since V is a generator in $\mathfrak{D}$, there is an exact sequence $0 \longrightarrow Y \xrightarrow{f} V^{(I)} \xrightarrow{g} X \longrightarrow 0$, where I is a set. If we apply the functor G we get the long exact sequence

$$\begin{array}{c} 0 \longrightarrow G(X) \longrightarrow G(V^{(I)}) \longrightarrow G(Y) \longrightarrow R^1G((X)) \longrightarrow \\ R^1G((V^{(I)})) \longrightarrow R^1G((Y)) \longrightarrow .... \end{array}$$

By assumption $R^iG((X)) = 0$ for any $i \geq 1$. Since $R^iG((V^{(I)})) = 0$, for any $i \geq 1$, $R^iG((Y)) = 0$ for any $i \geq 1$, by the exactness. Thus $Y \in^\perp T_G^{i\geqslant 1}$ and hence by Lemma 2.2, $G(Y) \in n\text{-}Copres(U)$. Since (F, G, V, U) is a Co-$*^n$-tuple, applying the functor F to the following sequence

$$0 \longrightarrow G(X) \longrightarrow G(V^{(I)}) \longrightarrow G(Y) \longrightarrow 0$$

we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & V^{(I)} & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow_{\delta'_Y} & & \downarrow_{\delta'_{V^{(I)}}} & & \downarrow_{\delta'_X} & & \\ 0 & \longrightarrow & FG(Y) & \longrightarrow & FG(V^{(I)}) & \longrightarrow & FG(X) & \longrightarrow & 0 \end{array}$$

Hence by Snake Lemma, δ'_X is an epimorphism and $Ker(\delta'_X) \cong Co\ker(\delta'_Y)$, since $\delta'_{V^{(I)}}$ is an isomorphism. Similarly δ'_Y is also an epimorphism. Thus, δ'_X is an isomorphism and therefore $X \cong FG(X) \in F(n\text{-}Copres(U))$. So $F(n\text{-}Copres(U)) = ^{\perp} T_G^{i \geqslant 1}$. □

Proposition 3.3. *Suppose that (F, G, V, U) be a Co-$*^n$-tuple. Then F preserves any exact sequence in n-$Copres(U)$.*

Proof. Let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be an exact sequence in $n\text{-}Copres(U)$. Applying the functor F we get the following long exact sequence

$$0 \longrightarrow F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \xrightarrow{\alpha} R^1F(Z) \longrightarrow ....$$

Thus we can get the following two exact sequences

$$0 \longrightarrow W \longrightarrow F(X) \longrightarrow Q \longrightarrow 0,$$

$$0 \longrightarrow F(Z) \longrightarrow F(Y) \longrightarrow W \longrightarrow 0,$$

where $Q = \operatorname{Im} \alpha$ and $W = \operatorname{Im} F(f)$. Applying the functor G to the last sequence we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow_{\delta_Y} & & \downarrow_{\delta_Z} & & & & \\ 0 & \longrightarrow & G(W) & \longrightarrow & GF(Y) & \longrightarrow & GF(Z) & \longrightarrow & R^1G(W) & \longrightarrow & 0 \end{array}.$$

It is clear by Proposition 3.1 that δ_Y and δ_Z are isomorphisms and $R^iG(F(Y)) = 0 = R^iG(F(Z))$, for any $i \geq 1$. By Snake Lemma, $X \cong G(W)$ and by the exactness $R^iG(W) = 0$ for any $i \geq 1$. Hence by Proposition 3.2, $W = F(D)$ for some $D \in n\text{-}Copres(U)$. Therefore

$$W = F(D) \cong F(GF(D)) = FG(F(D)) = FG(W) \cong F(X).$$

Hence $F(f)$ is an epimorphism, and therefore the induced sequence

$$0 \longrightarrow F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \longrightarrow 0$$

is exact. □

Theorem 3.2. *The following conditions are equivalent:*
(1) (F, G, V, U) is a Co-$*^n$-tuple.
(2) (i) U is co-F-small,
(ii) For any exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ with $X, Y \in n\text{-}Copres(U)$, we have $Z \in n\text{-}Copres(U)$ if and only if $0 \longrightarrow F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \longrightarrow 0$ is exact.

Proof. (1)⇒(2) The necessity follows from Proposition 3.3 and the sufficiency from a similar proof to that of (1)⇒(2) in Theorem 3.1.
(2)⇒(1) It follows from (2)⇒(1) in Theorem 3.1. □

Proposition 3.4. *Suppose that (F, G, V, U) is a Co-$*^n$-tuple. Then $n\text{-}Copres(U)$ is closed under extensions if and only if $n\text{-}Copres(U) \subseteq^{\perp} T_F^1 = \{X \in \mathfrak{C} : R^1F(X) = 0\}$.*

Proof. Suppose that $n\text{-}Copres(U)$ is closed under extensions. For any $X \in n\text{-}Copres(U)$ one can construct an exact sequence using the canonical maps to get an extension $0 \longrightarrow U \longrightarrow Y \longrightarrow X \longrightarrow 0$ of U by X. We have $Y \in n\text{-}Copres(U)$, by assumption. By Proposition 3.3, F preserves any exact sequence in $n\text{-}Copres(U)$, so applying F to the last exact sequence we get the exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(U) \longrightarrow 0,$$

thus by the exactness, $R^1F(X) = 0$, so $X \in^{\perp} T_F^1$ and hence $n\text{-}Copres(U) \subseteq^{\perp} T_F^1$. Conversely. For any extension $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, of X by Z, where $X, Z \in n\text{-}Copres(U)$, the induced sequence

$$0 \longrightarrow F(Z) \longrightarrow F(Y) \longrightarrow F(X) \longrightarrow 0,$$

is exact by assumption. According to Proposition 3.1, both δ_X and δ_Z are isomorphisms and $F(X), F(Z) \in^{\perp} T_G^{i\geq 1}$. Then it is clear that δ_Y is an isomorphism and $F(Y) \in^{\perp} T_G^{i\geq 1}$. Hence by Lemma 2.2, we have $Y \cong GF(Y) \in n\text{-}Copres(U)$. ☐

Theorem 3.3. *The following conditions are equivalent:*
(1) (F, G, V, U) is a Co-$*^n$-tuple.
(2) There is a duality

$$G :^{\perp} T_G^{i\geq 1} \rightleftarrows n\text{-}Copres(U) : F$$

Proof. (1)⇒(2) By Propositions 3.1 and Propositions 3.2.
(2)⇒(1) Since $V^{(I)} \in^{\perp} T_G^{i\geq 1}$, we get $F(U^I) \cong F(G(V)^I) \cong F(G(V^{(I)})) \cong FG(V^{(I)}) \cong V^{(I)} \cong F(U)^{(I)}$. So U is co-F-small. For any $X \in n\text{-}Copres(U)$, by assumption $X \cong G(F(X))$ and $F(X) \in^{\perp} T_G^{i\geq 1}$, thus $X \in (n+1)\text{-}Copres(U)$, by Lemma 2.2. So $n\text{-}Copres(U) = (n+1)\text{-}Copres(U)$. Now let $0 \longrightarrow X \xrightarrow{f} U^I \longrightarrow Y \longrightarrow 0$ be an exact sequence, with $Y \in (n+1)\text{-}Copres(U)$ and I a set. We can get the following exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \xrightarrow{F(f)} F(X) \longrightarrow Q \longrightarrow 0,$$

where $Q = \operatorname{Im}\alpha$, where $\alpha : F(X) \longrightarrow R^1F(Y)$. By using argument similar to that in Proposition 3.3, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & U^I & \longrightarrow & Y & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow_{\delta_{U^I}} & & \downarrow_{\delta_Y} & & & & \\ 0 & \longrightarrow & G(W) & \longrightarrow & GF(U^I) & \longrightarrow & GF(Y) & \longrightarrow & R^1G(W) & \longrightarrow & 0 \end{array},$$

where $W = \operatorname{Im} F(f)$, and we conclude that $Q = 0$, which means that we have an exact sequence $0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0$. Thus (F, G, V, U) is a Co-$*^n$-tuple. ☐

Proposition 3.5. *Suppose that U is a co-F-small . Assume that $n\text{-}Copres(U) = {}^{\perp}T_F^{i\geq 1}$. Then (F, G, V, U) is a Co-$*^n$-tuple.*

Proof. Let $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, be an exact sequence with $X \in n\text{-}Copres(U)$ and I a set. We can get the following long exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow R^1F(Y) \longrightarrow$$
$$R^1F(U^I) \longrightarrow R^1F(X) \longrightarrow ....$$

Note that $X, U^I \in n\text{-}Copres(U) =^{\perp} T_F^{i\geq 1}$, so by exactness, $R^iF(Y) = 0$, for every $i \geq 2$. Now $R^1F(Y) = 0$ if and only if $Y \in {}^{\perp}T_F^{i\geq 1} = n\text{-}Copres(U)$. So by Theorem 3.1 we get the desired result. ☐

4 Conclusion

We introduced the concept of Co-$*^n$-tuple of contravariant functors and give some characterizations as in Theorems 3.1, 3.2 and 3.3.

Competing interests

The author declares that no competing interests exist.

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