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Statistical Inference for Kumaraswamy Distribution Based on Generalized Order Statistics with Applications

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Abstract

In this paper, non-Bayesian and Bayesian approaches are used to obtain point and interval estimation of the shape parameters, the reliability and the hazard rate functions of the Kumaraswamy distribution. The estimators are obtained based on generalized order statistics. The symmetric and asymmetric loss functions are considered for Bayesian estimation. Also, maximum likelihood and Bayesian prediction for a new observation are found. The results have been specialized to Type II censored data and the upper record values. Comparisons are made between Bayesian and non-Bayesian estimates via Monte Carlo simulation. Moreover, the results are applied on real hydrological data.

Keywords: Kumaraswamy distribution; generalized order statistics; loss functions; Type-II censored data; upper records; maximum likelihood prediction; Bayesian prediction.

1. Introduction

[1,2] introduced the concept of generalized order statistics denoted by GOS to unify several concepts that have been used in statistics such as ordinary order statistics, record values, sequential order statistics, Pfeifer's record model and others. He presented his concepts of GOS by defining the joint density function of n random variables, which are called uniform generalized order statistics $U(r, n, \tilde{m}, k),\ r = 1, 2, \dots, m$ as follows

$$f^{U(1,n,\tilde{m},k),\dots,U(n,n,\tilde{m},k)}(u_1, \dots, u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right](1-u_n)^{k-1}, \quad (1)$$

where

$k \geq 1,\quad n \geq 1 \in N,\quad 0 \leq u_1 \leq \cdots \leq u_n < 1, \gamma_r = k + n - r + M_r, \gamma_r > 0,$
$M_r = \sum_{i=r}^{n} m_i,\quad m_1, \dots, m_{n-1} \in \mathcal{R}$ and $\tilde{m} = m_1, \dots, m_{n-1}, r \in \{1, \dots, r-1\}.$

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[1] used the quantile transformation $X(r,n,\tilde{m},k) = F^{-1}(U(r,n,\tilde{m},k))$ based on an arbitrary absolutely continuous distribution function $F(\mathrm{x})$ to define the joint probability density function (pdf) of the GOS $X(r,n,\tilde{m},k)$, $r = 1,\dots,n$ in the form

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) = \\ k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\left((1-F(x_i)\right)^{m_i} f(x_i)\right][1-F(x_n)]^{k-1}f(x_n)\,, \tag{2}$$

where

$$F^{-1}(0) \le x_1 \le \cdots \le x_n < F^{-1}(1).$$

Hence, the joint pdf of the first r GOS can be written as

$$f^{X(1,n,\tilde{m},k),\dots,X(r,n,\tilde{m},k)}(x_1,\dots,x_r) = \\ C_{r-1}\left[\prod_{i=1}^{r-1}\left((1-F(x_i)\right)^{m_i} f(x_i)\right][1-F(x_r)]^{k+n-r+M_r-1}f(x_r), \tag{3}$$

where

$$C_{r-1} = \prod_{j=1}^{r}\gamma_j, \quad r = 1,\dots,n,\ \ \gamma_j = k+n-j+M_j,\ \ \gamma_n = k.$$

If $m_1 = \cdots = m_{n-1} = m$ the joint pdf in (3) takes the form

$$f^{X(1,n,m,k),\dots,X(r,n,m,k)}(x_1,\dots,x_r) = \\ C_{r-1}\left[\prod_{i=1}^{r-1}\left((1-F(x_i)\right)^{m} f(x_i)\right][1-F(x_r)]^{\gamma_r-1}f(x_r), \tag{4}$$

where

$$C_{r-1} = \prod_{j=1}^{r}\gamma_j, \quad r = 1,\dots,n,\ \ \gamma_j = k+(n-j)(m+1),\ \ \gamma_n = k.$$

[3,4] showed that the well known pdf`s such as the normal, log-normal and Beta distributions do not fit well hydrological data like daily rain fall and daily stream flow. He developed the sine power pdf to fit up hydrological random processes which have lower and upper ends and which has a mode between them. [5] dropped the latter condition and proposed more general pdf for double bounded random processes with hydrological applications. The Kumaraswamy distribution is very similar to Beta distribution but has the key advantage of a closed-form cumulative distribution function (cdf) so it has been used only in simulation modeling and the estimation of models by simulation-based methods for 20 years. Although the Kumaraswamy distribution was introduced in 1980, the first theoretical investigation to it was presented by [6]. He derived an expression for the moments, studied the distribution's limiting distributions, introduced an analytical expression for the mean absolute deviation around the median as a function of the parameters of the distribution, established some bounds for this dispersion measure and for the variance and examined the relationship between the Kumaraswamy distribution and the Beta family of distributions.

[7] derived the distribution of single order statistics, the joint distribution of two order statistics and the distribution of the product and the quotient of two order statistics when the random variables are independent and identically Kumaraswamy-distributed.

[8] derived general formulas for L-moments and the moments of order statistics of the Kumaraswamy distribution. He studied the distribution's skewness and kurtosis properties. Moreover, he considered maximum likelihood estimation (ML) for the parameters of Kumaraswamy distribution. He compared between the Beta and the Kumaraswamy distributions from several points of view.

[9] obtained both the joint distribution and distributions of the product and the ratio of two GOS from Kumaraswamy distribution.

[10] obtained the classical estimators of one of the shape parameters of the Kumaraswamy distribution and compared these estimators according to their mean squared errors (MSEs). They got the Bayes estimators of the Kumaraswamy distribution for grouped and un-grouped data.

[11] used maximum likelihood and Bayesian approaches to obtain the estimators of the parameters and the future s-th record value from the Kumaraswamy distribution.

The generalized double-bounded, Kumaraswamy's pdf and cdf are given, respectively, by

$$f(\mathrm{x}) = \alpha\beta \mathrm{x}^{\alpha-1}(1-\mathrm{x}^{\alpha})^{\beta-1}, \tag{5}$$

and

$$F(\mathrm{x}) = 1-[1-\mathrm{x}^{\alpha}]^{\beta}, \qquad 0<\mathrm{x}<1, \alpha>0 \text{ and } \beta>0. \tag{6}$$

The reliability function (rf) and hazard rate function (hrf) are given, respectively, by

$$R(\mathrm{x}) = [1-\mathrm{x}^{\alpha}]^{\beta}, \tag{7}$$

and

$$h(\mathrm{x}) = \frac{\alpha\beta\mathrm{x}^{\alpha-1}}{1-\mathrm{x}^{\alpha}}. \tag{8}$$

In Section 2, ML estimators of the parameters, the rf and the hrf based on GOS are derived. Bayes point estimators and credible interval of the parameters, the rf and hrf based on GOS are given in Section 3. In Section 4 the one-sample and two-sample prediction are taken in consideration, respectively. Three applications are used in Section 5 to demonstrate how the proposed methods can be used in practice. In Section 6, Monte Carlo simulation study is performed to investigate and compare the methods of ML and Bayesian estimation and prediction.

2. Maximum Likelihood Estimation

Suppose that $X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k), k>0, \tilde{m}=(m_1,m_2,\ldots,m_{n-1}) \in \mathcal{R}^{n-1}$, $m_1,\ldots,m_{n-1} \in \mathcal{R}$ be the first r GOS drawn from Kumaraswamy population whose pdf and cdf are given by (5) and (6). Then the likelihood function is given by

$$\ell(\alpha,\beta;\ \underline{x}) \propto \alpha^r\beta^r \prod_{i=1}^{r} x_i^{\alpha} \prod_{i=1}^{r-1}(1-x_i^{\alpha})^{\beta(m_i+1)-1}\,(1-x_r^{\alpha})^{\beta\gamma_r-1}, \tag{9}$$

hence, the logarithm of the likelihood function is given by

$$\begin{aligned} L(\alpha,\beta;\ \underline{\mathrm{x}}) = {} & r\ln\alpha + r\ln\beta + \alpha\sum_{i=1}^{r}\ln x_i + \sum_{i=1}^{r-1}[\beta(m_i+1)-1]\ln(1-x_i^{\alpha}) \\ & +(\beta\gamma_r-1)\ln(1-x_r^{\alpha}). \end{aligned} \tag{10}$$

Differentiating the log-likelihood function in (10) with respect to α and β one can obtain

$$\frac{\partial L}{\partial\beta} = \frac{r}{\beta} + \sum_{i=1}^{r-1}(m_i+1)\ln(1-x_i^{\alpha}) + \gamma_r\ln(1-x_r^{\alpha}), \tag{11}$$

and

$$\frac{\partial L}{\partial\alpha} = \frac{r}{\alpha} + \sum_{i=1}^{r}\ln x_i - \sum_{i=1}^{r-1}[\beta(m_i+1)-1]\frac{x_i^{\alpha}\ln x_i}{(1-x_i^{\alpha})} - (\beta\gamma_r-1)\frac{x_r^{\alpha}\ln x_r}{(1-x_r^{\alpha})}. \tag{12}$$

Equating the derivatives (11) and (12) to zero, one can obtain the ML estimator of the parameter β which is given by

$$\hat{\beta}_{ML} = \frac{-r}{\left[\sum_{i=1}^{r-1}(m_i+1)\ln\left(1-x_i^{\hat{\alpha}}\right)+\gamma_r\ln(1-x_r^{\hat{\alpha}})\right]}, \tag{13}$$

and by substituting (13) in (12), the ML estimate of the parameter α is obtained numerically. Applying the properties of ML estimates, ML estimators of R(x) and h(x), based on GOS model, are derived. The Bayesian point estimation and credible interval of R(x) and h(x), based on GOS model, are given in Section 3.

$$\hat{R}_{ML}(\mathrm{x}) = \left[1-x^{\hat{\alpha}}\right]^{\hat{\beta}}, \tag{14}$$

and

$$\hat{h}_{ML}(\mathrm{x}) = \frac{\hat{\alpha}\hat{\beta}x^{\hat{\alpha}-1}}{1-x^{\hat{\alpha}}}. \tag{15}$$

The asymptotic variance - covariance matrix of the ML estimates for the two parameters α and β is

$$\tilde{I}^{-1} = \begin{bmatrix} \widetilde{\mathrm{var}}(\hat{\alpha}_{ML}) & \widetilde{\mathrm{cov}}(\hat{\alpha}_{ML},\hat{\beta}_{ML}) \\ \widetilde{\mathrm{cov}}(\hat{\alpha}_{ML},\hat{\beta}_{ML}) & \widetilde{\mathrm{var}}(\hat{\beta}_{ML}) \end{bmatrix} = \frac{1}{|I|}\left.\begin{bmatrix} -\frac{\partial^2 L}{\partial\beta^2} & \frac{\partial^2 L}{\partial\alpha\partial\beta} \\ \frac{\partial^2 L}{\partial\beta\partial\alpha} & -\frac{\partial^2 L}{\partial\alpha^2} \end{bmatrix}\right|_{\hat{\alpha}_{ML},\hat{\beta}_{ML}}, \tag{16}$$

with

$$\frac{\partial^2 L}{\partial \alpha^2} = \frac{-r}{\alpha^2} - \sum_{i=1}^{r-1}[\beta(m_i+1)-1][\frac{x_i^\alpha \ln(x_i)^2}{(1-x_i^\alpha)} + \frac{(x_i^\alpha)^2 \ln(x_i)^2}{(1-x_i^\alpha)^2}] + (\beta\gamma_r - 1)[\frac{x_r^\alpha \ln(x_r)^2}{(1-x_r^\alpha)} + \frac{(x_r^\alpha)^2 \ln(x_r)^2}{(1-x_r^\alpha)^2}], \tag{17}$$

$$\frac{\partial^2 L}{\partial \beta^2} = \frac{-r}{\beta^2}, \tag{18}$$

and

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = \frac{\partial^2 L}{\partial \beta \partial \alpha} = -\sum_{i=1}^{r-1}(m_i+1)\frac{x_i^\alpha \ln x_i}{(1-x_i^\alpha)} - \gamma_r \frac{x_r^\alpha \ln x_r}{(1-x_r^\alpha)}. \tag{19}$$

The asymptotic normality of the ML estimates can be used to compute the asymptotic confidence intervals for the parameters α and β, and become respectively as follows

$$\hat{\alpha}_{ML} \pm Z_{(1-\tau)/_2}\sqrt{\widehat{\mathrm{var}}(\hat{\alpha})}, \tag{20}$$

and

$$\hat{\beta}_{ML} \pm Z_{(1-\tau)/_2}\sqrt{\widehat{\mathrm{var}}(\hat{\beta})}, \tag{21}$$

where $Z_{(1-\tau)/_2}$ is a standard normal variate and τ is the confidence coefficient.

3. Bayesian Estimation

In this section, the Bayesian point and credible intervals estimation for the parameters of Kumaraswamy distribution based on GOS are derived.

Assuming that the parameters α and β are random variables with a joint bivariate prior density function that was used by [12] as,

$$g(\alpha, \beta) = g_1(\beta|\alpha) g_2(\alpha), \tag{22}$$

where

$$g_1(\beta|\alpha) = \frac{\alpha^{a+1}}{\Gamma(a+1)\omega^{a+1}}\beta^a e^{-\frac{\alpha\beta}{\omega}} \qquad , a > -1,\ \omega > 0, \tag{23}$$

and the prior of α is

$$g_2(\alpha) = \frac{\alpha^{\delta-1}}{\Gamma(\delta) b^\delta} e^{-\frac{\alpha}{b}}, \qquad b > 0, \delta > 0, \tag{24}$$

which is the gamma (δ, b) density. Substituting (23) and (24) in (22) one can obtain the bivariate prior density of α and β as follows

$$\pi(\alpha,\beta) \propto \alpha^{\delta+a}\beta^{a}e^{-\alpha\left(\frac{1}{b}+\frac{\beta}{\omega}\right)}, \tag{25}$$

where $a > -1, \delta, b$ and ω are positive real numbers.
The joint posterior distribution of α and β could be obtained using (9) and (25), as follows

$$\pi\left(\alpha,\beta|\underline{x}\right) \propto \ell\left(\alpha,\beta;\underline{\mathrm{x}}\right).\pi(\alpha,\beta), \tag{26}$$

hence,

$$\pi\left(\alpha,\beta|\underline{x}\right) = \frac{\alpha^{\delta+a+r}\beta^{a+r}e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}e^{-\beta[(\frac{\alpha}{\omega}-\sum_{\mathrm{i=1}}^{\mathrm{r-1}}(m_i+1)\ln\left(1-x_i^{\alpha}\right)-\gamma_r\ln\left(1-x_r^{\alpha}\right)]}}{\prod_{i=1}^{r}\left(1-x_i^{\alpha}\right)\Gamma(a+r+1)\psi(0,1,0,0)}, \tag{27}$$

where

$$\psi(c,d,h,f) = \int_0^{\infty}\frac{\alpha^{\delta+a+r+c}e^{-\alpha(h+\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}}{\prod_{i=1}^{r}\left(1-x_i^{\alpha}\right)[f+\frac{\alpha}{\omega}-\sum_{\mathrm{i=1}}^{\mathrm{r-1}}(m_i+1)\ln\left(1-x_i^{\alpha}\right)-\gamma_r\ln(1-x_r^{\alpha})]^{r+a+d}}d\alpha. \tag{28}$$

Bayes estimators are obtained based on four different types of loss functions, the squared error (SE) loss function (as a symmetric loss function), linear exponential LINEX, precautionary (P) and general entropy (GE) loss functions (as asymmetric loss functions).

3.1 Squared Error Loss Function

The SE loss function is a symmetric loss function and takes the form

$$L(\theta,\theta^*) = c(\theta-\theta^*)^{2},$$

where c denotes a constant and θ^* is an estimator. The Bayes estimator with respect to a quadratic loss function is the mean of the posterior distribution which takes the form

$$\theta_{SE}^* = E\left(\theta|\underline{\mathrm{x}}\right) = \int_{\Theta}\theta\, q\left(\theta|\underline{\mathrm{x}}\right)d\theta.$$

Hence, the Bayes estimators of the parameters α, β, the rf and the hrf under SE loss function are given respectively by

$$\alpha_{SE}^* = \frac{\psi(1,1,0,0)}{\psi(0,1,0,0)}, \tag{29}$$

$$\beta_{SE}^* = (r+a+1)\frac{\psi(0,2,0,0)}{\psi(0,1,0,0)}, \tag{30}$$

$$R_{SE}^*(x) = \frac{\psi(0,1,0,-\ln(1-x^{\alpha}))}{\psi(0,1,0,0)}, \tag{31}$$

and

$$h^*_{SE}(x) = \int_0^\infty \frac{x^{\alpha-1}(a+r+1)\alpha^{\delta+a+r+1}e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}}{(1-x^\alpha)\prod_{i=1}^{r}(1-x_i^\alpha)\left[\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}(m_i+1)\ln(1-x_i^\alpha)-\gamma_r\ln(1-x_r^\alpha)\right]^{r+a+2}\psi(0,1,0,0)}d\alpha. \tag{32}$$

3.2 Linear Exponential Loss Function

The linear-exponential (LINEX) is an asymmetric loss function defined as

$$L(\theta,\theta^*) = e^{\upsilon(\theta-\theta^*)} - \upsilon(\theta-\theta^*) - 1,$$

under LINEX loss function, the Bayes estimator θ^* of θ is given by

$$\theta^*_{LNX} = \frac{-1}{\upsilon}\ln E_\theta\left(e^{-\upsilon\theta}\middle|\underline{\mathrm{x}}\right),$$

where E_θ stands for posterior expectation. Therefore, the Bayes estimators of the parameters α, β, the rf and hrf based on GOS under LINEX loss function are given by

$$\alpha^*_{LNX} = \frac{1}{-\upsilon}\ln\frac{\psi(0,1,\upsilon,0)}{\psi(0,1,0,0)}, \tag{33}$$

$$\beta^*_{LNX} = \frac{1}{-\upsilon}\ln\frac{\psi(0,1,0,\upsilon)}{\psi(0,1,0,0)}, \tag{34}$$

$$R^*_{LNX}(x) = \frac{1}{-\upsilon}\ln E\left(e^{-\upsilon R(x)}\middle|\underline{\mathrm{x}}\right), \tag{35}$$

where

$$E\left(e^{-\upsilon R(x)}\middle|\underline{\mathrm{x}}\right) = \int_0^\infty\int_0^\infty \frac{\alpha^{\delta+a+r}\beta^{a+r}e^{-\upsilon(1-x^\alpha)^\beta}e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}e^{-\beta\left(\upsilon+\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}(m_i+1)\ln(1-x_i^\alpha)-\gamma_r\ln(1-x_r^\alpha)\right)}}{\prod_{i=1}^{r}(1-x_i^\alpha)\Gamma(a+r+1)\psi(0,1,0,0)}d\beta d\alpha,$$

and

$$h^*_{LNX}(x) = \frac{1}{-\upsilon}\ln E\left(e^{-\upsilon h(x)}\middle|\underline{\mathrm{x}}\right), \tag{36}$$

where

$$E\left(e^{-\upsilon h(x)}\middle|\underline{\mathrm{x}}\right) = \int_0^\infty \frac{e^{-\upsilon(\frac{x^{\alpha-1}}{(1-x^\alpha)})}\alpha^{\delta+a+r+1}e^{-\alpha(\upsilon+\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}}{\prod_{i=1}^{r}(1-x_i^\alpha)\left[\upsilon+\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}(m_i+1)\ln(1-x_i^\alpha)-\gamma_r\ln(1-x_r^\alpha)\right]^{r+a+1}\psi(0,1,0,0)}d\alpha.$$

3.3 Precautionary Loss Function

[13] introduced an alternative asymmetric loss function, and also presented a general class of P loss functions as a special case. These loss functions approach infinitely to the origin to prevent

under estimation, thus giving conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when underestimation may lead to serious consequences.

A very useful and simple asymmetric P loss function is

$$L(\theta^*,\theta)=\frac{(\theta^*-\theta)^2}{\theta^*\theta}.$$

The Bayes estimator under P loss function can be obtained by solving the following equation

$$(\theta_p^*)^2=\frac{E\left(\theta\middle|\underline{x}\right)}{E\left(\theta^{-1}\middle|\underline{x}\right)},$$

hence, the Bayes estimators for the parameters, the rf and the hrf based on GOS under the P loss function are given by

$$\alpha_p^*=\sqrt{\frac{\psi(-1,1,0,0)}{\psi(1,1,0,0)}}, \tag{37}$$

$$\beta_p^*=\sqrt{\frac{\psi(0,0,0,0)}{(a+r)(a+r+1)\psi(0,2,0,0)}}, \tag{38}$$

$$R_P^*(x)=\sqrt{\frac{\psi(0,1,0,\ln(1-x^\alpha))}{\psi(0,1,0,-\ln(1-x^\alpha))}}, \tag{39}$$

and

$$h_P^*(x)=\sqrt{\frac{h_{SE}^*(x)}{\Psi_*}}, \tag{40}$$

where

$$\Psi_*=\int_0^\infty\frac{(1-x^\alpha)\alpha^{\delta+a+r-1}e^{-\alpha(\frac{1}{b}-\sum_{i=1}^r\ln x_i)}}{x^{(\alpha-1)}(a+r+1)\prod_{i=1}^r(1-x_i^\alpha)\left[\frac{\alpha}{\omega}-\sum_{\mathrm{i}=1}^{\mathrm{r}-1}(m_i+1)\ln\left(1-x_i^\alpha\right)-\gamma_r\ln(1-x_r^\alpha)\right]^{r+a}\psi(0,1,0,0)}d\alpha.$$

3.4 General Entropy Loss Function

In many practical situations, it appears to be realistic to express the loss in terms of ratio $\frac{\theta^*}{\theta}$. [14] pointed out a generalization of the entropy loss function as an asymmetric loss function

$$L(\theta,\theta^*)\propto\left({\theta^*}/{\theta}\right)^q-q\log\left({\theta^*}/{\theta}\right)-1,$$

whose minimum occurs at $\theta^*=\theta$ and the shape parameter $q=1$, (see [15]). When $q>0$, a positive error $(\theta^*>\theta)$ causes serious consequences than a negative error. The Bayes estimator θ_{GE}^* of θ under the GE loss is

$$\theta_{GE}^{*} = [E_{\theta}(\theta^{-q})]^{-1/q},$$

Therefore, the Bayes estimators of the parameters α, β, the rf and the hrf under GE loss function could be written as

$$\alpha_{GE}^{*} = \left(\frac{\psi(-q,1,0.0)}{\psi(0,1,0,0)}\right)^{\frac{-1}{q}}, \tag{41}$$

$$\beta_{GE}^{*} = \left(\frac{\Gamma(a+r+1-q)}{\Gamma(a+r+1)}\frac{\psi(0,1-q,0.0)}{\psi(0,1,0,0)}\right)^{\frac{-1}{q}}, \tag{42}$$

$$R_{GE}^{*}(x) = \left(\frac{\psi(0,1,0,q\ln(1-x^{\alpha}))}{\psi(0,1,0,0)}\right)^{\frac{-1}{q}}, \tag{43}$$

and

$$h_{GE}^{*}(x) = \left(\frac{\Gamma(a+r-q+1)\Omega(x_i,x_r)}{\Gamma(a+r+1)}\right)^{\frac{-1}{q}}, \tag{44}$$

where

$$\Omega(x_i,x_r) = \int_0^{\infty} \frac{\alpha^{\delta+a+r-q}\mathrm{x}^{-q(\alpha-1)}e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}}{(1-\mathrm{x}^{\alpha})^{-q}\prod_{i=1}^{r}(1-x_i^{\alpha})\psi(0,1,0,0)\left[\frac{\alpha}{\omega}-\sum_{\mathrm{i}=1}^{\mathrm{r}-1}(m_i+1)\ln(1-x_i^{\alpha})-\gamma_r\ln(1-x_r^{\alpha})+q\ln(1-\mathrm{x}^{\alpha})\right]^{r+a-q+1}}\,d\alpha. \tag{45}$$

The Bayesian analog to the confidence interval is called a credibility interval. In general, $(\mathrm{L}(\underline{\mathrm{x}}),\mathrm{U}(\underline{\mathrm{x}}))$ is a $100(1-\tau)\%$ credibility interval of θ if

$$P[L(\underline{\mathrm{x}})|\underline{\mathrm{x}} < \theta < U(\underline{\mathrm{x}})|\underline{\mathrm{x}}] = \int_{L(\underline{\mathrm{x}}),}^{U(\underline{x})}\pi(\theta|\underline{x})\,d\theta = 1-\tau.$$

Since, the posterior distribution is given by (27), then a $100(1-\tau)\%$ credibility interval for α based on GOS is$(\mathrm{L}(\underline{\mathrm{x}}),\mathrm{U}(\underline{\mathrm{x}}))$, where

$$\begin{aligned} P[\alpha > L(\underline{\mathrm{x}})|\underline{\mathrm{x}}] &= \int_{L(\underline{\mathrm{x}})}^{\infty}\frac{\alpha^{\delta+a+r}e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}}{\psi(0,1,0,0)\prod_{i=1}^{r}(1-x_i^{\alpha})\left[\frac{\alpha}{\omega}-\sum_{\mathrm{i}=1}^{\mathrm{r}-1}(m_i+1)\ln(1-x_i^{\alpha})-\gamma_r\ln(1-x_r^{\alpha})\right]^{r+a+1}}\,d\alpha \\ &= 1-\frac{\tau}{2}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} P[\alpha > U(\underline{\mathrm{x}})|\underline{\mathrm{x}}] &= \int_{U(\underline{\mathrm{x}})}^{\infty}\frac{\alpha^{\delta+a+r}e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)}}{\psi(0,1,0,0)\prod_{i=1}^{r}(1-x_i^{\alpha})\left[\frac{\alpha}{\omega}-\sum_{\mathrm{i}=1}^{\mathrm{r}-1}(m_i+1)\ln(1-x_i^{\alpha})-\gamma_r\ln(1-x_r^{\alpha})\right]^{r+a+1}}\,d\alpha \\ &= \frac{\tau}{2}, \end{aligned} \tag{47}$$

also, a $100(1-\tau)\%$ credibility interval for β based on GOS is $(\mathrm{L}(\underline{\mathrm{x}}),\mathrm{U}(\underline{\mathrm{x}}))$, where

$$P\left[\beta > L(\underline{\mathrm{x}})\middle|\underline{\mathrm{x}}\right] = \beta^{a+r} \times \int_{L(\underline{\mathrm{x}})}^{\infty}\int_{0}^{\infty} \frac{\alpha^{\delta+a+r} e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)} e^{-\beta\left(\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}(m_i+1)\ln(1-x_i^{\alpha})-\gamma_r\ln(1-x_r^{\alpha})\right)}}{\prod_{i=1}^{r}(1-x_i^{\alpha})\Gamma(a+r+1)\psi(0,1,0,0)} d\alpha\, d\beta = 1-\frac{\tau}{2}, \tag{48}$$

and

$$P\left[\beta > L(\underline{\mathrm{x}})\middle|\underline{\mathrm{x}}\right] = \beta^{a+r} \times \int_{L(\underline{\mathrm{x}})}^{\infty}\int_{0}^{\infty} \frac{\alpha^{\delta+a+r} e^{-\alpha(\frac{1}{b}-\sum_{i=1}^{r}\ln x_i)} e^{-\beta\left(\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}(m_i+1)\ln(1-x_i^{\alpha})-\gamma_r\ln(1-x_r^{\alpha})\right)}}{\prod_{i=1}^{r}(1-x_i^{\alpha})\Gamma(a+r+1)\psi(0,1,0,0)} d\alpha\, d\beta = \frac{\tau}{2}, \tag{49}$$

4. Prediction

In this study, one and two-sample prediction are considered. ML prediction and Bayesian prediction for a new observation are used.

4.1 One-Sample Prediction

Suppose that the first r GOS $X(1,n,\widetilde{m},k),\dots,X(r,n,\widetilde{m},k)$ have been observed and we wish to predict the future GOS $X(r+1,n,\widetilde{m},k),\dots,X(n,n,\widetilde{m},k)$.

Let $Y_s = X_s$, $s = r+1, r+2, \dots, n$, $m_i = m$, $\forall\, i = 1,\dots,r$. The conditional density function of the s$^{\text{th}}$ future GOS, given the first r, is given by

$$f\left(y_s\middle|\alpha,\beta;\underline{\mathrm{x}}\right) = \begin{cases} \frac{k^{s-r}}{(s-r-1)!}\left[h_m(F(y_s)-h_m(F(x_r)\right]^{s-r-1}\frac{[1-F(y_s)]^{k-1}f(y_s)}{[1-F(y_s)]^{k}}, & m=-1,\\[2ex] \frac{C_{s-1}}{C_{r-1}(s-r-1)!}\left[h_m(F(y_s)-h_m(F(x_r)\right]^{s-r-1}\frac{[1-F(y_s)]^{\gamma_s-1}f(y_s)}{[1-F(y_s)]^{\gamma_{r+1}}}, & m\neq -1, \end{cases} \tag{50}$$

where

$$h_m(x) = \begin{cases} \frac{-(1-x)^{m+1}}{m+1}, & m \neq -1\\ -\ln(1-x), & m=-1 \end{cases}, \tag{51}$$

$$C_{\ell-1} = \prod_{j=1}^{\ell}\gamma_j,\ \ell = s, r, \tag{52}$$

and

$$\gamma_j = k + (n-j)(m+1),\ \ \forall\, j \in \{1,2,\dots,n-1\}. \tag{53}$$

Therefore, the marginal density function of Y_s will be obtained for two different cases; for $m \neq -1$ and for $m = -1$.
Substituting (5) and (6) in (51) one can obtain,

$$h_m(F(x)) = \begin{cases} \frac{-1}{m+1}(1-x^{\alpha})^{\beta(m+1)}, m \neq -1 \\ -\beta \ln(1-x^{\alpha}), \quad m=-1 \end{cases}, \tag{54}$$

then, the conditional density function of the s^{th} future GOS, given the first r, from Kumaraswamy distribution could be obtained from (54) and (50). Therefore,

$$f_1(y_s|\alpha,\beta;\underline{\mathrm{x}}) = \begin{cases} A\alpha\beta y_s^{\alpha-1}\sum_{j=0}^{s-r-1} a_j\,(1-{y_s}^{\alpha})^{\beta V_j^*-1}(1-{x_r}^{\alpha})^{\beta V_j},\ m\neq -1, \\ \\ \frac{k^{s-r}y_s^{\alpha-1}}{\Gamma(s-r)}\alpha\beta^{s-r}(1-{x_r}^{\alpha})^{-k\beta}(1-{y_s}^{\alpha})^{k\beta-1} \\ \times\sum_{j=0}^{s-r-1} a_j[-\ln(1-x_r^{\alpha})]^{s-r-j-1}\,[-\ln(1-y_s^{\alpha})]^j, m=-1, \end{cases} \tag{55}$$

where

$$A = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}.(\frac{-1}{m+1})^{s-r-1}, \tag{56}$$

$$a_j = (-1)^{s-r-1-j}\binom{s-r-1}{j}, \tag{57}$$

$$V_j^* = j(m+1)+\gamma_s, \tag{58}$$

and

$$V_j = (m+1)(s-r-1-j)-\gamma_{r+1}. \tag{59}$$

4.1.1 Maximum likelihood prediction based on one-sample prediction

The ML prediction can be obtained using the conditional density function of the s^{th} future GOS, given the first r, which is given by (55) after replacing the shape parameters α and β by their ML estimates $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$.

The ML predictive estimator of the future GOS $Y_s, s=r+1, r+2, ...$, given the first r GOS, is given by

$$\hat{y}_{s_{(ML)}} = \int\quad y_s f(y_s|\hat{\alpha}_{ML},\hat{\beta}_{ML};\underline{\mathrm{x}})\,dy_s, \tag{60}$$

Substituting (55) in (60), one obtains

$$\hat{y}_{s_{(ML)}} = \begin{cases} \int_{x_r}^{1} A\alpha\hat{\beta}_{ML}y_s^{\alpha}\sum_{j=0}^{s-r-1} a_j\,(1-{y_s}^{\alpha})^{\beta V_j^*-1}(1-{x_r}^{\alpha})^{\beta V_j}\,dy_s, m\neq -1, \\ \\ \int_{x_r}^{1}\frac{k^{s-r}y_s^{\alpha}}{\Gamma(s-r)}\hat{\beta}_{ML}{}^{s-r}\left(1-{x_r}^{\hat{\alpha}_{ML}}\right)^{-\hat{\beta}_{ML}k} \\ \sum_{j=0}^{s-r-1} a_j[-\ln(1-x_r^{\hat{\alpha}_{ML}})]^{s-r-j-1}\,[-\ln\left(1-y_s^{\hat{\alpha}_{ML}}\right)]^j, \quad m=-1, \end{cases} \tag{61}$$

where A, a_j, V_j^* and V_j are given by (56), (57), (58) and (59).

A $(1-\tau)$ 100 % maximum likelihood predictive bounds (MLPB) for the future observation Y_s, $s=r+1,r+2,\ldots,n$, given the first r, such that

$P\left[L\left(\underline{\mathrm{x}}\right) \leq Y_s \leq U\left(\underline{\mathrm{x}}\right)\right] = 1-\tau$, are as follows

$$P\left[Y_s > L\left(\underline{\mathrm{x}}\right) | \underline{\mathrm{x}}\right] = \int_{L\left(\underline{\mathrm{x}}\right)}^{1} f_1\left(y_s | \hat{\alpha}_{ML}, \hat{\beta}_{ML}; \underline{\mathrm{x}}\right) dy_s = 1-\frac{\tau}{2}, \tag{62}$$

and

$$P\left[Y_s > U\left(\underline{\mathrm{x}}\right) | \underline{\mathrm{x}}\right] = \int_{U\left(\underline{\mathrm{x}}\right)}^{1} f_1\left(y_s | \hat{\alpha}_{ML}, \hat{\beta}_{ML}; \underline{\mathrm{x}}\right) dy_s = \frac{\tau}{2}. \tag{63}$$

Substituting (55) in (62) and (63) the MLPB for the future observation Y_s, $s=r+1,r+2,\ldots,n$, given the first r based on GOS, are given by

$$P\left[Y_s > L\left(\underline{\mathrm{x}}\right) | \underline{\mathrm{x}}\right] = \begin{cases} \int_{L(x)}^{1} A\hat{\alpha}_{ML}\hat{\beta}_{ML} y_s^{\hat{\alpha}_{ML}-1} \\ \times \sum_{j=0}^{s-r-1} a_j \left(1-{y_s}^{\hat{\alpha}_{ML}}\right)^{\hat{\beta}_{ML}V_j^*-1} \left(1-{x_r}^{\hat{\alpha}_{ML}}\right)^{\hat{\beta}_{ML}V_j} dy_s, \qquad m \neq -1, \\ \int_{L(x)}^{1} \frac{k^{s-r} y_s^{\hat{\alpha}_{ML}-1}}{\Gamma(s-r)} \hat{\alpha}_{ML} {\hat{\beta}_{ML}}^{s-r} \left(1-{x_r}^{\hat{\alpha}_{ML}}\right)^{-\hat{\beta}_{ML}k} \left(1-{y_s}^{\hat{\alpha}_{ML}}\right)^{k\hat{\beta}_{ML}-1} \\ \times \sum_{j=0}^{s-r-1} a_j [-\ln(1-x_r^{\hat{\alpha}_{ML}})]^{s-r-j-1} \left[-\ln\left(1-y_s^{\hat{\alpha}_{ML}}\right)\right]^j dy_s, \quad m=-1, \end{cases} \tag{64}$$

and

$$P\left[Y_s > U\left(\underline{\mathrm{x}}\right) | \underline{\mathrm{x}}\right] = \begin{cases} \int_{U(x)}^{1} A\hat{\alpha}_{ML}\hat{\beta}_{ML} y_s^{\hat{\alpha}_{ML}-1} \\ \times \sum_{j=0}^{s-r-1} a_j \left(1-{y_s}^{\hat{\alpha}_{ML}}\right)^{\hat{\beta}_{ML}V_j^*-1} \left(1-{x_r}^{\hat{\alpha}_{ML}}\right)^{\hat{\beta}_{ML}V_j}, dy_s \qquad m \neq -1, \\ \int_{U(x)}^{1} \frac{k^{s-r} y_s^{\hat{\alpha}_{ML}-1}}{\Gamma(s-r)} \hat{\alpha}_{ML} {\hat{\beta}_{ML}}^{s-r} \left(1-{x_r}^{\hat{\alpha}_{ML}}\right)^{-\hat{\beta}_{ML}k} \left(1-{y_s}^{\hat{\alpha}_{ML}}\right)^{k\hat{\beta}_{ML}-1} \\ \times \sum_{j=0}^{s-r-1} a_j [-\ln(1-x_r^{\hat{\alpha}_{ML}})]^{s-r-j-1} \left[-\ln\left(1-y_s^{\hat{\alpha}_{ML}}\right)\right]^j dy_s, \quad m=-1. \end{cases} \tag{65}$$

4.1.2 Bayesian prediction based on one-sample prediction

The Bayesian predictive density (BPD) function of the future Y_s could be obtained by applying the following equation

$$h\left(y_s | \underline{x}\right) = \int_0^\infty \int_0^\infty f\left(y_s | \alpha, \beta; \underline{\mathrm{x}}\right) . \pi\left(\alpha, \beta | \underline{\mathrm{x}}\right) d\beta \, d\alpha, \tag{66}$$

Therefore, the BPD of the future Y_s, $s=r+1,r+2,\ldots$ given the first r is given by substituting (27) and (55) in (66) to be

$$h_1(y_s|\underline{x}) = \begin{cases} A\int_0^\infty \frac{\alpha^{\delta+a+r+1}e^{-\alpha[\frac{1}{b}-\sum_{i=1}^r \ln x_i - \ln y_s]}}{y_s(1-y_s^\alpha)\prod_{i=1}^r(1-x_i^\alpha)\psi(0,1,0,0)} \\ \times \frac{(a+r+1)}{\sum_{j=1}^{s-r-1} a_j[\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}\ln(1-x_i)-(\gamma_r+V_j)\ln(1-x_r{}^\alpha)-V_j^*\ln(1-y_s{}^\alpha)]^{r+a+2}}\,d\alpha, \qquad m \neq -1, \\ \\ \int_0^\infty \frac{k^{s-r}\alpha^{\delta+a+r+1}e^{-\alpha\left[\frac{1}{b}-\sum_{i=1}^r \ln x_i - \ln y_s\right]}}{y_s{}^\alpha(1-y_s{}^\alpha)\mathrm{B}(s-r,a+r+1)\prod_{i=1}^r(1-x_i^\alpha)\psi_1(0,1,0,0)[\frac{\alpha}{\omega}+(k-\gamma_r)\ln(1-x_r{}^\alpha)-k\ln(1-y_s{}^\alpha)]^{a+s+1}} \\ \times\sum_{j=1}^{s-r-1}\binom{s-r-1}{j}[\ln(1-x_r^\alpha)]^j[\ln(1-y_s^\alpha)]^{s-r-j-1}\,d\alpha, \qquad m=-1, \end{cases} \tag{67}$$

hence, the Bayes predictive estimator (BPE) of the future Y_s, $s = r+1, r+2, \ldots$ given the first r under SE loss function is given by

$$y_{s(SE)}^* = \begin{cases} A\int_{x_r}^1\int_0^\infty \frac{\alpha^{\delta+a+r+1}e^{-\alpha[\frac{1}{b}-\sum_{i=1}^r \ln x_i - \ln y_s]}}{(1-y_s^\alpha)\prod_{i=1}^r(1-x_i^\alpha)\psi(0,1,0,0)} \\ \times \frac{(a+r+1)}{\sum_{j=0}^{s-r-1} a_j[\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}\ln(1-x_i)-(\gamma_r+V_j)\ln(1-x_r{}^\alpha)-V_j^*\ln(1-y_s{}^\alpha)\;]^{r+a+2}}\,d\alpha\,dy_s, \quad m \neq -1, \\ \\ \int_{x_r}^1\int_0^\infty \frac{y_s{}^{1-\alpha}k^{s-r}\alpha^{\delta+a+r+1}e^{-\alpha\left[\frac{1}{b}-\sum_{i=1}^r \ln x_i - \ln y_s\right]}}{(1-y_s{}^\alpha)\mathrm{B}(s-r,a+r+1)\prod_{i=1}^r(1-x_i^\alpha)\psi_1(0,1,0,0)[\frac{\alpha}{\omega}+(k-\gamma_r)\ln(1-x_r{}^\alpha)-k\ln(1-y_s{}^\alpha)]^{a+s+1}} \\ \times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}[\ln(1-x_r^\alpha)]^j[\ln(1-y_s^\alpha)]^{s-r-j-1}\,d\alpha\,dy_s, \qquad m=-1, \end{cases} \tag{68}$$

and the BPE of the future $Y_s, s = r+1, r+2, \ldots$ given the first r under LINEX loss function is given by

$$y_{s(LINEX)}^* = \frac{-1}{\upsilon}\ln E(e^{-\upsilon y_s}|\underline{\mathrm{x}}),$$

where

$$E\left(e^{-\upsilon y_s}|\underline{\mathrm{x}}\right) = \begin{cases} A\int_{x_r}^1\int_0^\infty \frac{\alpha^{\delta+a+r+1}e^{-\upsilon y_s}e^{-\alpha[\frac{1}{b}-\sum_{i=1}^r \ln x_i - \ln y_s]}}{(1-y_s^\alpha)\prod_{i=1}^r(1-x_i^\alpha)\psi(0,1,0,0)} \\ \times \frac{(a+r+1)}{\sum_{j=0}^{s-r-1} a_j[\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}\ln(1-x_i)-(\gamma_r+V_j)\ln(1-x_r{}^\alpha)-V_j^*\ln(1-y_s{}^\alpha)]^{r+a+2}}\,d\alpha\,dy_s, \qquad m \neq -1, \\ \\ \int_{x_r}^1\int_0^\infty \frac{k^{s-r}\alpha^{\delta+a+r+1}e^{-\upsilon y_s}e^{-\alpha\left[\frac{1}{b}-\sum_{i=1}^r \ln x_i - \ln y_s\right]}}{y_s{}^\alpha(1-y_s{}^\alpha)\mathrm{B}(s-r,a+r+1)\prod_{i=1}^r(1-x_i^\alpha)\psi_1(0,1,0,0)[\frac{\alpha}{\omega}+(k-\gamma_r)\ln(1-x_r{}^\alpha)-k\ln(1-y_s{}^\alpha)]^{a+s+1}} \\ \times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}[\ln(1-x_r^\alpha)]^j[\ln(1-y_s^\alpha)]^{s-r-j-1}\,d\alpha\,dy_s, \qquad m=-1. \end{cases} \tag{69}$$

A $(1-\tau)$ 100 % Bayesian predictive bounds (BPB) for the future observation Y_s, $s = r+1, r+2, \ldots, n$, given the first r, such that $P\left[L(\underline{\mathrm{x}}) \le Y_s \le U(\underline{\mathrm{x}})\right] = 1-\tau$, are as follows

$$P\left[Y_s > L(\underline{x})\middle|\underline{x}\right] = \int_{L(\underline{x})}^{1} h(y_s|\underline{x})dy_s = 1 - \frac{\tau}{2}, \tag{70}$$

and

$$P\left[Y_s > U(\underline{x})\middle|\underline{x}\right] = \int_{U(\underline{x})}^{1} h(y_s|\underline{x})dy_s = \frac{\tau}{2}. \tag{71}$$

Substituting (67) in (70) and (71) one can obtain

$$P\left[Y_s > L(\underline{x})\middle|\underline{x}\right] = \begin{cases} A\int_{L(\underline{x})}^{1}\int_0^\infty \frac{\alpha^{\delta+a+r+1}e^{-\alpha[\frac{1}{b}-\sum_{i=1}^{r}\ln x_i-\ln y_s]}}{y_s(1-y_s^\alpha)\prod_{i=1}^{r}(1-x_i^\alpha)\psi(0,1,0,0)} \\ \times\frac{(a+r+1)}{\sum_{j=0}^{s-r-1}a_j[\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}\ln(1-x_i)-(\gamma_r+V_j)\ln(1-x_r{}^\alpha)-V_j^*\ln(1-y_s{}^\alpha)]^{r+a+2}}\,d\alpha\, dy_s, & m\neq-1, \\ \int_{L(\underline{x})}^{1}\int_0^\infty \frac{k^{s-r}\alpha^{\delta+a+r+1}e^{-\alpha\left[\frac{1}{b}-\sum_{i=1}^{r}\ln x_i-\ln y_s\right]}}{y_s{}^\alpha(1-y_s{}^\alpha)\mathrm{B}(s-r,a+r+1)\prod_{i=1}^{r}(1-x_i^\alpha)\psi_1(0,1,0,0)[\frac{\alpha}{\omega}+(k-\gamma_r)\ln(1-x_r{}^\alpha)-k\ln(1-y_s{}^\alpha)]^{a+s+1}} \\ \times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}[\ln(1-x_r^\alpha)]^j[\ln(1-y_s^\alpha)]^{s-r-j-1}\,d\alpha\, dy_s, & m=-1, \end{cases} \tag{72}$$

and

$$P\left[Y_s > U(\underline{x})\middle|\underline{x}\right] = \begin{cases} A\int_{U(\underline{x})}^{1}\int_0^\infty \frac{\alpha^{\delta+a+r+1}e^{-\alpha[\frac{1}{b}-\sum_{i=1}^{r}\ln x_i-\ln y_s]}}{y_s(1-y_s^\alpha)\prod_{i=1}^{r}(1-x_i^\alpha)\psi(0,1,0,0)} \\ \times\frac{(a+r+1)}{\sum_{j=0}^{s-r-1}a_j[\frac{\alpha}{\omega}-\sum_{i=1}^{r-1}\ln(1-x_i)-(\gamma_r+V_j)\ln(1-x_r{}^\alpha)-V_j^*\ln(1-y_s{}^\alpha)]^{r+a+2}}\,d\alpha\, dy_s, & m\neq-1 \\ \int_{U(\underline{x})}^{1}\int_0^\infty \frac{k^{s-r}\alpha^{\delta+a+r+1}e^{-\alpha\left[\frac{1}{b}-\sum_{i=1}^{r}\ln x_i-\ln y_s\right]}}{y_s{}^\alpha(1-y_s{}^\alpha)\mathrm{B}(s-r,a+r+1)\prod_{i=1}^{r}(1-x_i^\alpha)\psi_1(0,1,0,0)[\frac{\alpha}{\omega}+(k-\gamma_r)\ln(1-x_r{}^\alpha)-k\ln(1-y_s{}^\alpha)]^{a+s+1}} \\ \times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}[\ln(1-x_r^\alpha)]^j[\ln(1-y_s^\alpha)]^{s-r-j-1}\,d\alpha\, dy_s, & m=-1. \end{cases} \tag{73}$$

4.2 Two-Sample Prediction

Let $X(1,n,m,k),\dots,X(r,n,m,k)$ be a GOS of size n from Kumaraswamy distribution and suppose $Y(1,N,M,K),\dots,Y(R,N,M,K)$, $K>0, M\in R$ is a second unobserved GOS of size N. The density function of the GOS Y_s is given by,

$$f(y_s|\alpha,\beta) = \begin{cases} \frac{C_{s-1}^*}{(s-1)!}[1-F(y_s)]^{\gamma_s{}^*-1}f(y_s)[g_M(F(y_s))]^{s-1}, & M\neq-1, \\ \frac{k^s}{(s-1)!}[1-F(y_s)]^{k-1}f(y_s)[g_M(F(y_s))]^{s-1}, & M=-1, \end{cases} \tag{74}$$

where

$$g_M(x) = \begin{cases} \frac{[1-(1-x)^{M+1}]}{M+1}, & M \neq -1, \\ -\ln(1-x), & M = -1, \end{cases} \tag{75}$$

and

$$C_{s-1}^* = \prod_{\ell=1}^{s} \gamma_\ell^*, \gamma_\ell^* = K + (N-\ell)(M+1). \tag{76}$$

Substituting (5), (6) in (75), then

$$\left[g_M\big((F(y_s)\big)\right]^{s-1} = \begin{cases} \frac{1}{(M+1)^{s-1}}\left\{\left[1-(1-y_s^{\alpha})^{\beta(M+1)}\right]\right\}^{s-1}, M \neq -1, \\ \beta^{s-1}[-\ln(1-y_s^{\alpha})]^{s-1}, \qquad M = -1. \end{cases} \tag{77}$$

For the future sample of size n, let Y_s , denote the s^{th} ordered lifetime, $1 \leq s \leq n$, then the density function of the GOS Y_s from Kumaraswamy distribution is obtained by substituting (5), (6) and (77) in (74) where $0 < y_s < 1$, hence

$$f_2(y_s|\alpha,\beta) = \begin{cases} \eta\alpha\beta y_s^{\alpha-1}\sum_{j=0}^{s-1} c_j (1-y_s^{\alpha})^{\beta t_j - 1} & , M \neq -1, \\ \frac{k^s}{(s-1)!}\alpha\beta^s y_s^{\alpha-1}(1-y_s^{\alpha})^{\beta k-1}(-\ln(1-y_s^{\alpha}))^{s-1}, M = -1, \end{cases} \tag{78}$$

where

$$\eta = \frac{C_{s-1}^*}{(s-1)!(M+1)^{s-1}},$$

and

$$c_j = (-1)^j \binom{s-1}{j} \quad \text{and} \quad t_j = [\gamma_s^* + j(M+1)].$$

4.2.1 Maximum likelihood prediction based on two-sample prediction

The point and interval ML prediction for the parameters α and β is obtained based on GOS. The results are specialized to Type II censoring and upper records.

The ML predictive estimator of the future observation Y_s, $1 \leq s \leq n$, could be obtained by substituting (78) in (60)

$$\hat{y}_{s(SE)} = \begin{cases} \eta\hat{\beta}_{ML}\sum_{j=0}^{s-1} c_j \,\mathrm{B}\left(\frac{1}{\hat{\alpha}_{ML}}+1, \hat{\beta}_{ML} t_j\right), & M \neq -1, \\ \frac{k^s}{(s-1)!}\hat{\alpha}_{ML}\hat{\beta}_{ML}{}^{s}\int_0^1 y_s^{\hat{\alpha}_{ML}}(1-y_s^{\hat{\alpha}_{ML}})^{\hat{\beta}_{ML}k-1}(-\ln(1-y_s^{\hat{\alpha}_{ML}}))^{s-1}\, dy_s, M = -1, \end{cases} \tag{79}$$

the ML predictive bounds for the GOS $Y_s, 1 \leq s \leq N$ are given by substituting (78) in (70) and (71) therefore,

$$P(Y_s > L(x)|\underline{x}) = \begin{cases} \int_{L(x)}^{1} \eta \hat{\alpha}_{ML} \hat{\beta}_{ML} y_s^{\hat{\alpha}_{ML}-1} \sum_{j=0}^{s-1} c_j (1 - y_s^{\hat{\alpha}_{ML}})^{\hat{\beta}_{ML} t_j - 1} \, dy_s, & M \neq -1, \\ \frac{k^{s-r}}{(s-r-1)!} \hat{\alpha}_{ML} \hat{\beta}_{ML}^{s} \\ \times \int_{L(x)}^{1} y_s^{\hat{\alpha}_{ML}} (1 - y_s^{\hat{\alpha}_{ML}})^{k\hat{\beta}_{ML}-1} (-\ln(1 - y_s^{\hat{\alpha}_{ML}}))^{s-1} \, dy_s, & M = -1, \end{cases} \tag{80}$$

and

$$P(Y_s > U(x)|\underline{x}) = \begin{cases} \int_{U(x)}^{1} \eta \hat{\alpha}_{ML} \hat{\beta}_{ML} y_s^{\hat{\alpha}_{ML}-1} \sum_{j=0}^{s-1} c_j (1 - y_s^{\hat{\alpha}_{ML}})^{\hat{\beta}_{ML} t_j - 1} \, dy_s, & M \neq -1, \\ \frac{k^{s-r}}{(s-r-1)!} \hat{\alpha}_{ML} \hat{\beta}_{ML}^{s} \\ \times \int_{U(x)}^{1} y_s^{\hat{\alpha}_{ML}} (1 - y_s^{\hat{\alpha}_{ML}})^{k\hat{\beta}_{ML}-1} (-\ln(1 - y_s^{\hat{\alpha}_{ML}}))^{s-1} \, dy_s, & M = -1. \end{cases} \tag{81}$$

4.2.2 Bayesian prediction based on two-sample prediction

The BPD function of the future observation Y_s, $1 \leq s \leq N$ is given by using (27) and (78) as follows

$$h_2(y_s|\underline{x}) = \begin{cases} \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^{r} \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1)\ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \sum_{j=0}^{s-1} c_j t_j \ln(1-y_s^\alpha)]^{a+r+2}} \\ \times \frac{(a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha)\prod_{i=1}^{r}(1-x_i^\alpha)\psi(0,1,0,0)} d\alpha, & M \neq -1, \\ \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^{r} \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1)\ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - k\ln(1-y_s^\alpha)]^{a+r+2}} \\ \frac{(-\ln(1-y_s^\alpha))^{s-1}(a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha)\prod_{i=1}^{r}(1-x_i^\alpha)\psi(0,1,0,0)} d\alpha, & M = -1, \end{cases} \tag{82}$$

hence, the BPE of the future Y_s, $1 \leq s \leq N$ under SE loss function is given by

$$\hat{y}_{s(SE)} = \begin{cases} \int_0^1 \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^{r} \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1)\ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \sum_{j=0}^{s-1} c_j t_j \ln(1-y_s^\alpha)]^{a+r+2}} \\ \times \frac{(a+r+1)\alpha^{\delta+a+r+1}}{(1-y_s^\alpha)\prod_{i=1}^{r}(1-x_i^\alpha)\psi(0,1,0,0)} d\alpha dy_s, & M \neq -1, \\ \int_0^1 \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^{r} \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1)\ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - k\ln(1-y_s^\alpha)]^{a+r+2}} \\ \frac{(-\ln(1-y_s^\alpha))^{s-1}(a+r+1)\alpha^{\delta+a+r+1}}{(1-y_s^\alpha)\prod_{i=1}^{r}(1-x_i^\alpha)\psi(0,1,0,0)} d\alpha dy_s, & M = -1, \end{cases} \tag{83}$$

the BPE of the future Y_s, $1 \leq s \leq N$ under LINEX loss function is given by

$$y^*_{s(LINEX)} = \frac{-1}{\upsilon} \ln E(e^{-\upsilon y_s} | \underline{\mathrm{x}})$$

where

$$E\left(e^{-\upsilon y_s} \middle| \underline{\mathrm{x}}\right) = \begin{cases} \int_0^1 \int_0^\infty \frac{e^{-\upsilon y_s} e^{-\alpha(\frac{1}{b} - \sum_{i=1}^r \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1) \ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \sum_{j=0}^{s-1} c_j t_j \ln(1-y_s^\alpha)]^{a+r+2}} \\ \times \frac{(a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha) \prod_{i=1}^r (1-x_i^\alpha) \psi(0,1,0,0)} d\alpha dy_s, & M \neq -1, \\ \int_0^1 \int_0^\infty \frac{e^{-\upsilon y_s} e^{-\alpha(\frac{1}{b} - \sum_{i=1}^r \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1) \ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \mathrm{k}\ln(1-y_s^\alpha)]^{a+r+2}} \\ \times \frac{(-\ln(1-y_s^\alpha))^{s-1} (a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha) \prod_{i=1}^r (1-x_i^\alpha) \psi(0,1,0,0)} d\alpha dy_s, & M = -1. \end{cases} \tag{84}$$

A $(1-\tau)$ 100 % BPB for the future observationY_s, such that $P\left[L(\underline{\mathrm{x}}) \leq Y_s \leq U(\underline{\mathrm{x}})\right] = 1 - \tau$ are

$$P\left(Y_s > L(x) \middle| \underline{\mathrm{x}}\right) = \begin{cases} \int_{L(x)}^1 \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^r \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1) \ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \sum_{j=0}^{s-1} c_j t_j \ln(1-y_s^\alpha)]^{a+r+2}} \\ \times \frac{(a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha) \prod_{i=1}^r (1-x_i^\alpha) \psi(0,1,0,0)} d\alpha \, dy_s, & M \neq -1, \\ \int_{L(x)}^1 \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^r \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1) \ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \mathrm{k}\ln(1-y_s^\alpha)]^{a+r+2}} \\ \frac{(-\ln(1-y_s^\alpha))^{s-1} (a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha) \prod_{i=1}^r (1-x_i^\alpha) \psi(0,1,0,0)} d\alpha \, dy_s, & M = -1, \end{cases} \tag{85}$$

and

$$P\left(Y_s > U(x) \middle| \underline{\mathrm{x}}\right) = \begin{cases} \int_{U(x)}^1 \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^r \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1) \ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \sum_{j=0}^{s-1} c_j t_j \ln(1-y_s^\alpha)]^{a+r+2}} \\ \frac{(a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha) \prod_{i=1}^r (1-x_i^\alpha) \psi(0,1,0,0)} \, d\alpha \, dy_s, & M \neq -1, \\ \int_{U(x)}^1 \int_0^\infty \frac{e^{-\alpha(\frac{1}{b} - \sum_{i=1}^r \ln x_i - \ln y_s)}}{[(\frac{\alpha}{\omega} - \sum_{i=1}^{r-1} (m_i+1) \ln(1-x_i^\alpha) - \gamma_r \ln(1-x_r^\alpha) - \mathrm{k}\ln(1-y_s^\alpha)]^{a+r+2}} \\ \frac{(-\ln(1-y_s^\alpha))^{s-1} (a+r+1)\alpha^{\delta+a+r+1}}{y_s(1-y_s^\alpha) \prod_{i=1}^r (1-x_i^\alpha) \psi(0,1,0,0)} \, d\alpha \, dy_s, & M = -1. \end{cases} \tag{86}$$

5. Some Applications

In this section, three hydrological real data sets are analyzed to demonstrate how the proposed methods can be used in practice. To check the validity of the fitted model, the Kolmogorov–Smirnov and chi–squared goodness of fit tests are performed through Easy Fit 5.5 Software.

Application 1:

The first application is a real data set obtained from the Shasta Reservoir in California, USA deals with the monthly water capacity data and were taken for the month of February from 1991 to 2010, (see [11]). The maximum capacity of the reservoir is 4552000 acre-foot (AF). In order to apply the real data to Kumaraswamy distribution the data were transformed to the interval [0, 1] by dividing the capacities over the capacity of the reservoir. The actual and transformed data of water capacity of Shasta reservoir in California from 1991 to 2010 is shown in Table 1 in the appendix.

Application 2:

The second application is the annual capacity of Naser Lake during the flood time from 1968 to 2010 obtained from [16] which is given in Table 2. Naser Lake is located in the lower Nile River Basin at the border between Egypt and Sudan. The total capacity of the lake is 162.3×10^9 Cubic meter (m^3) at its highest water level. The lake was created from the late 1960 to the 1968 together with the construction of the Aswan High Dam upstream of the old Aswan Dam, about 5 km south of the city of Aswan.

Application 3:

The third application is the annual water level behind the High Dam during the flood time from 1968 to 2010 obtained from [16] which is shown in Table 2. The highest water level of the Dam is 182 meter (m) above the mean sea level.

The Kolmogorov–Smirnov and Chi-Squared tests showed that the observations follow Kumaraswamy distribution. Easy Fit 5.5 software was used for fitting the observations. The p values are given, respectively, by 0.24465323, 0.63544848 and 0.6323368. Tables 3 and 6 show ML and Bayes estimates of the parameters, the rf and hrf for the real data based on Type II censoring and upper records as special cases of GOS by setting $m = 0$, $k = 1$ and $m = -1$, $k = 1$, respectively. The confidence and credible intervals for the parameters α and β based on Type II censoring and upper records are given in Tables 4 and 7. Table 5 shows ML and Bayes predictive estimates and bounds for the first future observations based on Type II censoring under one-sample and two-sample prediction. Table 8 shows ML and Bayes predictive estimates and bounds for the first future observation based on upper records under one-sample and two-sample prediction.

Remarks

It is clear from the p values given in each case that the model fits the data very well. The ML and Bayes estimates values are very close to each others for each data set. In most cases, the length of the credible intervals are shorter than the confidence intervals for both of the parameters α and β. Both of ML and Bayes predictive estimates of the future observations are very close to the value of the actual observation. In general, the lengths of the Bayesian predictive intervals are shorter than the ML predictive ones.

6. Simulation Study

In this section, a numerical example is given to illustrate the results obtained on basis of generated data from Kumaraswamy distribution. Also, to compare how the results are close for both methods; ML and Bayesian estimations. In order to get a ML and Bayes predictors, all computations are performed using Mathcad 14.

Table 9 shows the ML and Bayes estimates of the parameters, rf, hrf, their biases, relative absolute biases, estimated risks (ERs), relative mean square errors (MSEs) and variances based on Type II censoring where $N = 5000$ is the number of repetitions, $n = 50$, is the sample size, and $r = 45$ is the number of survival units, $\delta = 9$, $\omega = 0.09$, $a = 2$ and $b = 0.9$ are the values of the hyper parameters. Table 12 shows the ML and Bayes estimates of the parameters, the rf, the hrf, their biases, relative absolute biases, ERs, relative MSEs and variances based on upper records where $N = 2000$ is the number of repetitions, $n = 500$ is the sample size, $r = 7$ is the number of records, $R = 1043$ is the number of samples that have r records, $a = 7$, $b = 0.1$, $\delta = 8$, and $\omega = 0.1$ are the values of the hyper parameters. Tables 10 and 13 show confidence and credible intervals for the parameters α and β based on Type II censoring and upper records, respectively. Table 11 shows the ML and Bayesian predictive estimates and bounds for the first future observation based on Type II censoring under one-sample and two-sample prediction. Table 14 shows the ML and Bayes predictive estimates and bounds for the first future observation based on upper records under one-sample and two-sample prediction.

Concluding Remarks

1. Based on Type II censoring, it is clear from Table 9 that the ERs for the estimates of the parameters α , β and hrf under GE loss function have the less values than the corresponding ERs under the SE loss function then the corresponding ERs under the LINEX loss function then the corresponding ERs under the P loss function, and finally the ML ones but the ERs of the estimates of the rf under the P loss function has the less value than the corresponding ERs, then the corresponding ERs under SE loss function, then the corresponding ERs under the GE loss function after that the ML ERs and finally the corresponding ERs under the LINEX loss function.
2. Based on upper records, it has been noticed, from Table 12, that the ERs risks for the estimates of the parameters α and β and for the hrf under GE loss function have the less values than the corresponding ERs, then the P loss function comes next after that the SE loss function then the LINEX loss function, and finally the ML ERs. The ERs risks of the estimates of the rf under the GE loss function have the less value than the corresponding ERs, then ML ERs comes next, then LINEX loss function after that the SE and finally the P loss function.
3. Both of ML and Bayes predictive estimates of the future observation are very close to the value of the actual observations and the length of the Bayesian predictive intervals are shorter than the ML predictive ones. [Tables 11 and 14].

4. In most cases, the length of the credible intervals are shorter than the confidence intervals for both of the parameters α and β. [Tables 10 and 13]
5. Results perform better when n gets larger.

General Conclusion

In this study, ML and Bayes estimators for the shape parameters, rf and hrf of Kumaraswamy distribution, based on GOS, are obtained. ML and Bayesian prediction for a new observation from Kumaraswamy distribution, based on GOS, are derived. The Bayesian estimation is derived under four types of loss functions. The results are applied based on Type II censored data and upper record values as special cases from GOS. Monte Carlo simulation is used to construct the comparisons between Bayesian and non-Bayesian estimates. Moreover, the results are applied on real hydrological data.

In general, the length of the credible interval is shorter than the confidence intervals. Both of ML and Bayes predictive estimates of the future observations are very close to the value of the actual observation. In most cases, the lengths of the Bayes predictive intervals are shorter than the ML predictive ones. The Bayes estimates of the parameters and hrf based on Type II censoring under GE loss function have the smallest ERs. But the ERs of the estimates of the rf based on Type II censoring under the P loss function has the smallest ERs. The Bayes estimates of the parameters, rf and hrf based on upper records under GE loss function have the smallest ERs. We suggest using the Bayesian approach under GE loss function for estimating the parameters of Kumaraswamy distribution.

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Competing Interests

Authors have declared that no competing interests exist.

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APPENDIX

Table 1. Actual and transformed data of water capacity of Shasta Reservoir in California from 1991 to 2010

Year	Capacity (AF)	Ratio to total capacity	Year	Capacity (AF)	Ratio to total capacity
1991	1542838	0.338936	2001	3495969	0.768007
1992	1966077	0.431915	2002	3839544	0.843485
1993	3459209	0.759932	2003	3584283	0.787408
1994	3298496	0.724626	2004	3868600	0.849868
1995	3448519	0.757583	2005	3168056	0.69597
1996	3694201	0.811556	2006	3834224	0.842316
1997	3574861	0.785339	2007	3772193	0.828689
1998	3567220	0.78366	2008	2641041	0.580194
1999	3712733	0.815627	2009	1960458	0.430681
2000	3857423	0.847413	2010	3380147	0.742563

Table 2. Actual and transformed data of water level behind the High Dam and capacity of Nasser Lake during the flood time from 1968 to 2010

Year	Water level (m)	Ratio to maximum level	Capacity of the lake (Billion m^3)	Ratio to total capacity
1968/1969	151.1	0.83021	39.97	0.24627
1969/1970	153.83	0.84521	46.43	0.28607
1970/1971	159.68	0.87736	61.36	0.37806
1971/1972	162.49	0.89280	69.82	0.43019
1972/1973	158.2	0.86923	57.24	0.35268
1973/1974	161.71	0.88851	66.7	0.41096
1974/1975	165.60	0.90989	80.06	0.49328
1975/1976	172.43	0.94741	108.37	0.66771
1976/1977	171.70	0.94340	105.05	0.64725
1977/1978	172.52	0.94791	108.84	0.67060
1978/1979	173.04	0.95076	111.3	0.68576
1979/1980	171.27	0.94104	103.12	0.63536
1980/1981	171.13	0.94027	102.49	0.63148
1981/1982	170.36	0.93604	99.15	0.61090
1982/1983	165.87	0.91137	81.03	0.49926
1983/1984	163.60	0.89890	72.94	0.44941
1984/1985	156.37	0.85917	51.46	0.31706
1985/1986	157.23	0.86390	53.7	0.33086
1986/1987	154.65	0.84972	47.27	0.29125
1987/1988	151.70	0.83351	40.67	0.25058
1988/1989	164.41	0.90335	75.78	0.46691
1989/1990	163.77	0.89983	73.52	0.45298

(Continued) Table 2

Year	Water level (m)	Ratio to maximum level	Capacity of the lake (billion m^3)	Ratio to total capacity
1990/1991	162.5	0.89285	69.25	0.42667
1991/1992	163.98	0.90098	74.23	0.45736
1992/1993	167.45	0.92005	87.06	0.53641
1993/1994	169.64	0.932087	96.05	0.59180
1994/1995	172.34	0.94692	108	0.66543
1995/1996	172.76	0.94923	109.97	0.67757
1996/1997	175.48	0.96417	123.8	0.76278
1997/1998	174.75	0.96016	120	0.73937
1998/1999	174.75	0.96016	120	0.73937
1999/2000	175.79	0.96587	125.41	0.77270
2000/2001	175.85	0.96620	125.72	0.77461
2001/2002	175.7	0.96538	124.94	0.76980
2002/2003	175.14	0.96230	122.03	0.75187
2003/2004	172.06	0.94538	106.68	0.65730
2004/2005	169.59	0.93181	95.84	0.59051
2005/2006	168.65	0.92664	91.86	0.56598
2006/2007	173.42	0.95285	113.2	0.69747
2007/2008	174.8	0.960439	120.26	0.74097
2008/2009	173.3	0.95219	112.6	0.69377
2009/2010	169.79	0.93291	96.7	0.59581

Table 3. ML and Bayes estimates of the parameters, the reliability and the hazard rate functions for the real data based on Type II censoring

	Estimate	ML	Bayesian					
			SE	LINEX		P	GE	
Shasta Reservoir	$\hat{\alpha}$	3.40026	1.1637	$\upsilon = 0.5$	1.15446	1.12227	$q = 0.1$	1.11943
	$\hat{\beta}$	5.64983	0.54158		0.54016	0.52823		0.58964
	$\hat{R}(t)$	0.93388	0.71942		0.7188	0.71481		0.71447
	$\hat{h}(t)$	0.78081	1.01883		1.01464	0.60113		0.98691
Aswan High Dam	$\hat{\alpha}$	16.74034	3.19258	$\upsilon = 0.2$	3.15845	3.13867	$q = 0.1$	3.11225
	$\hat{\beta}$	3.1543	0.45389		0.45341	0.44856		1.38785
	$\hat{R}(t)$	0.99997	0.94418		0.94002	0.94388		0.93955
	$\hat{h}(t)$	9.64624×10^{-4}	0.36733		0.36168	0.03725		0.34103
Nasser Lake	$\hat{\alpha}$	3.99842	1.01589	$\upsilon = 0.2$	1.39187	1.00355	$q = 0.5$	1.38328
	$\hat{\beta}$	6.24748	1.39565		1.01341	1.38191		1.38067
	$\hat{R}(t)$	0.66787	0.95574		0.61415	0.61149		0.95296
	$\hat{h}(t)$	5.76047	0.61495		0.61112	0.96846		1.72369

Table 4. Confidence and credible intervals of the parameters for the real data based on Type II censoring

	Estimate	Method	L	U	Length
Shasta Reservoir	$\hat{\alpha}$	**ML**	4.05215	4.97511	0.92296
		Bayesian	0.38516	0.8235	0.43834
	$\hat{\beta}$	**ML**	1.12623	3.1672	2.04097
		Bayesian	1.39941	2.83843	1.43901
The High Dam	$\hat{\alpha}$	**ML**	13.66676	19.81392	6.14716
		Bayesian	8.52279	10.40727	1.88448
	$\hat{\beta}$	**ML**	2.18877	4.11983	1.93106
		Bayesian	1.33264	1.69111	0.35848
Nasser Lake	$\hat{\alpha}$	**ML**	3.02107	4.17347	1.1524
		Bayesian	0.9504	1.40309	0.45269
	$\hat{\beta}$	**ML**	3.3744	6.4051	3.0307
		Bayesian	1.2958	1.8339	0.5381

Table 5. ML and Bayes predictive estimates and bounds for the first future observation based on Type II censoring under one and two-sample prediction

Method	Data	x_s	Method	Interval prediction			Point prediction		
				95%			ML	Bayesian	
				L	U	Length		SE	LINEX
One-sample prediction	Shasta Reservoir	x_{20}	ML	0.72771	0.98463	0.25693	0.90679	0.97983	0.855002
			Bayesian	0.81286	0.99138	0.17851			
	Aswan High Dam	x_{42}	ML	0.93758	0.99714	0.05957	0.97642	0.90512	0.99106
			Bayesian	0.95901	0.99581	0.0368			
	Naser Lake		ML	0.88469	0.95	0.06531	0.78722	0.78567	0.92033
			Bayesian	0.76853	0.85476	0.08623			
Two-sample prediction	Shasta Reservoir	x_1	ML	0.6226	0.96249	0.3399	0.83208	0.92725	0.8495
			Bayesian	0.61994	0.99958	0.37963			
	Aswan High Dam		ML	0.74941	0.97803	0.22862	0.8961	0.87863	0.91617
			Bayesian	0.63112	0.9937	0.36258			
	Naser Lake		ML	0.23134	0.83808	0.60675	0.55997	0.80116	0.8114
			Bayesian	0.30831	0.99809	0.68978			

Table 6. ML and Bayes estimates of the parameters, the reliability and the hazard rate functions for the real data based on upper records

	Estimate	ML	Bayesian					
			SE	LINEX		P	GE	
Shasta Reservoir	$\hat{\alpha}$	2.87938	0.56821	$v = .5$	0.5512	0.75379	$q = .1$	0.54524
	$\hat{\beta}$	7.11741	2.92655		2.5132	2.83783		2.79546
	$\hat{R}(t)$	0.99064	0.40095		0.37096	0.73462		0.32607
	$\hat{h}(t)$	0.2709	2.16492		6.41408	3.52966		0.86183
Aswan High Dam	$\hat{\alpha}$	15.25355	0.58792	$v = 2$	0.570302	0.5724	$q = .2$	0.57098
	$\hat{\beta}$	12.27203	2.77572		2.456	2.70662		2.70067
	$\hat{R}(t)$	0.94806	0.01892		0.01851	2.7962510^{-4}		7.67210^{-3}
	$\hat{h}(t)$	1.16486	10.01844		6.82524	9.74035		0.07711
Nasser Lake	$\hat{\alpha}$	2.03326	0.6376	$v = 1$	0.61568	0.60342	$q = .2$	0.58708
	$\hat{\beta}$	12.1715	2.26025		2.04275	2.14823		2.09643
	$\hat{R}(t)$	0.89291	0.52777		0.50311	0.0804		0.42221
	$\hat{h}(t)$	2.31374	4.71637		2.9797	5.23795		1.0127

Table 7. Confidence and credible intervals of the parameters for the real data based on upper records

	Estimate	Method	L	U	Length
Shasta Reservoir	$\hat{\alpha}$	**ML**	2.50455	3.25421	0.74966
		Bayesian	0.34569	0.86736	0.52199
	$\hat{\beta}$	**ML**	3.31752	10.9173	7.59977
		Bayesian	0.34569	0.86736	0.52199
The High Dam	$\hat{\alpha}$	**ML**	10.81512	19.69199	8.87687
		Bayesian	1.712175	4.12684	2.41466
	$\hat{\beta}$	**ML**	10.98379	13.56028	2.57649
		Bayesian	0.359898	0.89125	0.53135
Nasser Lake	$\hat{\alpha}$	**ML**	0.3789	3.68763	3.30872
		Bayesian	1.1104523	3.86483	2.75438
	$\hat{\beta}$	**ML**	1.1445	22.05399	20.90989
		Bayesian	0.3141846	1.154021	0.8398

Table 8. ML and Bayes predictive estimates and bounds of the future observation for real data based on upper records under one and two-sample prediction

Method	Data	x_s	Method	Interval prediction 95% L	U	Length	Point prediction ML	Bayesian SE	LINEX
One-sample prediction	Shasta Reservoir	x_7	ML	0.84813	0.92055	0.07242	0.87085	0.867202	0.86574
			Bayesian	0.84797	0.91384	0.06587			
	Aswan High Dam	x_{11}	ML	0.96607	0.98464	0.01857	0.97198	0.971042	0.97094
			Bayesian	0.96603	0.98286	0.01683			
	Nasser Lake		ML	0.77318	0.82975	0.05658	0.78954	0.785467	0.78539
			Bayesian	0.77304	0.81857	0.04553			
Two-sample prediction	Shasta Reservoir	x_1	ML	0.70867	0.95469	0.24602	0.8563	0.95994	0.95952
			Bayesian	0.85063	0.99917	0.14854			
	Aswan High Dam		ML	0.78218	0.98348	0.20131	0.91239	0.95895	0.95854
			Bayesian	0.85088	0.99884	0.14797			
	Nasser Lake		ML	0.54267	0.82143	0.27876	0.69283	0.92988	0.92917
			Bayesian	0.79618	0.99451	0.19834			

Table 9. ML and Bayes estimates of the parameters, the reliability , the hazard rate functions and their biases, relative absolute biases, estimated risks, relative mean square errors and variances based on Type II censoring

$R = 5000, n = 50, r = 45, \alpha = 1.1, \beta = 0.9, \delta = 9, \omega = 0.09, a = 2, b = 0.9$

Method of estimation		estimate	Average	ER	Relative MSE	variance	bias	Relative absolute bias
ML		$\hat{\alpha}$	1.16508	0.06439	0.23069	0.06016	0.06508	0.05916
		$\hat{\beta}$	0.96268	0.05182	0.25293	0.04789	0.06268	0.06964
		$\hat{R}(t)$	0.92941	7.86678×10^{-4}	0.03022	7.85263×10^{-4}	1.18943×10^{-3}	1.28142×10^{-3}
		$\hat{h}(t)$	0.83145	0.05135	0.26527	0.05083	-0.02279	0.02668
Bayesian	SE	$\hat{\alpha}$	1.04417	0.02041	0.12987	0.01729	-0.05583	0.05076
		$\hat{\beta}$	0.7618	0.04504	0.2358	0.02594	-0.01382	0.15355
		$\hat{R}(t)$	0.92449	6.36359×10^{-4}	0.02718	6.2244×10^{-4}	-3.73085×10^{-3}	4.01936×10^{-3}
		$\hat{h}(t)$	0.81102	0.0331	0.21297	0.03123	-0.04322	0.05059
	LINEX $\nu = 0.4$	$\hat{\alpha}$	1.04176	0.02061	0.13051	0.01722	0.05825	0.05295
		$\hat{\beta}$	0.76075	0.04653	0.23969	0.02714	-0.13925	0.15472
		$\hat{R}(t)$	0.9273	1.10701×10^{-3}	0.03584	1.10626×10^{-3}	8.70187×10^{-3}	9.37482×10^{-3}
		$\hat{h}(t)$	0.80563	0.03449	0.217308	0.03216	-0.04861	0.05690
	P	$\hat{\alpha}$	1.03612	0.02094	0.13155	0.01686	-0.06388	0.05808
		$\hat{\beta}$	0.75288	0.04985	0.24809	0.02891	-0.14712	0.16347
		$\hat{R}(t)$	0.92491	4.86699×10^{-4}	0.02377	4.7575×10^{-4}	-3.08551×10^{-3}	3.56441×10^{-4}
		$\hat{h}(t)$	0.643442	0.03819	0.22876	0.12094	0.06999	0.081933
	GE $q = -1$	$\hat{\alpha}$	1.06929	0.01572	0.114163	0.014825	-0.030736	0.027942
		$\hat{\beta}$	0.850493	0.017963	0.148917	0.015512	-0.049507	0.055007
		$\hat{R}(t)$	0.921401	6.41399×10^{-4}	0.027284	5.949404×10^{-4}	-6.81609×10^{-3}	7.3432×10^{-3}
		$\hat{h}(t)$	0.831445	0.030681	0.20504	0.0301615	-0.022794	0.026683

Table 10. Confidence and credible intervals for the parameters α and β based on Type II censoring

	Method	L	U	Length
$\hat{\alpha}$	ML	0.79163	1.53902	0.74739
	Bayesian	0.81636	1.09053	0.27416
$\hat{\beta}$	ML	0.68101	1.24527	0.56425
	Bayesian	0.1.19364	1.56363	0.36966

Table 11. ML and Bayes predictive estimates and bounds for the first future observation based on Type II censoring under one and two-sample prediction

Method	x_s	Method	Interval prediction 95% L	U	Length	Point prediction ML	Bayesian SE	LINEX	
One-sample	x_{46}	ML	0.90301	0.95263	0.04962	0.91968	0.79827	$\upsilon = 1$	0.86068
		Bayesian	0.91674	0.96068	0.04394				
Two-sample	x_1	ML	0.05802	0.97745	0.91942	0.55698	0.98899	$\upsilon = -2$	0.90713
		Bayesian	0.53261	0.98182	0.44921				

Table 12. ML and Bayes estimates of the parameters, the reliability ,the hazard rate functions, biases, relative absolute biases, estimated risks, relative mean square errors and variances based on upper records

$R = 2000, n = 500, R = 1043, r = 7, \alpha = 1.1, \beta = 0.9, a = 7, b = 0.1, \delta = 8, \omega = 0.1, R(t) = 0.92822, h(t) = 0.85424, t = 0.1$

Method of estimation		Estimate	Average	ER	Relative MSE	variance	bias	Relative absolute bias
ML		$\hat{\alpha}$	1.9592	4.25305	1.87481	3.51393	0.85972	0.78156
		$\hat{\beta}$	1.00664	0.03957	0.22102	0.0282	0.10664	0.11849
		$\hat{R}(t)$	0.83299	5.48356×10^{-3}	0.01208	2.14086×10^{-3}	0.05501	0.05927
		$\hat{h}(t)$	1.61733	0.64929	0.91032	0.19532	-0.65512	0.7669
Bayesian	SE	$\hat{\alpha}$	0.85722	0.01349	0.10559	4.0792×10^{-3}	-0.2428	0.22071
		$\hat{\beta}$	0.99702	0.03638	0.21191	0.02696	0.09702	0.1078
		$\hat{R}(t)$	0.83176	0.01149	0.01238	2.18786×10^{-3}	-0.09646	0.10392
		$\hat{h}(t)$	1.54029	0.60592	0.69347	0.16166	-0.66653	0.76283
	LINEX $v = -0.1$	$\hat{\alpha}$	0.86002	0.06171	0.22584	4.12273×10^{-3}	-0.23998	0.21816
		$\hat{\beta}$	1.00179	0.037793	0.2164	0.02757	0.10179	0.1131
		$\hat{R}(t)$	0.83244	0.01133	0.01221	$.16102\times 10^{-3}$	-0.09577	0.10318
		$\hat{h}(t)$	1.57752	0.70032	0.81982	0.1772	-0.72328	0.84669
	P	$\hat{\alpha}$	0.82507	6.47152×10^{-3}	0.07313	3.86354×10^{-3}	-0.27493	0.24994
		$\hat{\beta}$	0.95107	0.02661	0.18126	0.02401	0.05107	0.05674
		$\hat{R}(t)$	0.77974	0.03566	0.03842	0.01361	-0.14848	0.15996
		$\hat{h}(t)$	1.51533	0.56839	0.65051	0.15678	0.64157	0.73426
	GE $q = 0.2$	$\hat{\alpha}$	0.81927	5.68737×10^{-3}	0.06856	3.82835×10^{-3}	-0.28073	0.25521
		$\hat{\beta}$	0.94312	0.02537	0.17699	0.02351	0.04312	0.04791
		$\hat{R}(t)$	0.82111	0.01422	0.01532	2.74943×10^{-3}	-0.10711	0.11539
		$\hat{h}(t)$	1.05452	0.083	0.09717	0.04289	-0.20028	0.23445

Table 13. Confidence and credible intervals for the parameters based on upper records

	Method	L	U	Length
$\hat{\alpha}$	ML	0.69312	3.28151	2.58839
	Bayesian	0.5599803	1.880402	1.32042
$\hat{\beta}$	Method	L	U	Length
	ML	0.36738	2.68317	2.31579
	Bayesian	1.7510333	2.162277	0.41124

Table 14. ML and Bayesian predictive estimates and bounds for the first future observation based on upper records under one and two-sample prediction

Method	x_s	Method	Interval prediction 95%			Point prediction			
			L	U	Length	ML	Bayesian SE	Bayesian LINEX	
One-sample	x_7	ML	0.9123	0.9725575	0.06026	0.97064	0.97277	$v = .1$	0.975951
		Bayesian	0.96236	0.99242	0.03006				
Two-sample	x_1	ML	0.50374	0.93783	0.43409	0.76684	0.907721	$v = .1$	0.9842
		Bayesian	0.69438	0.99928	0.3049				

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