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Research Article

Holographic Description of Noncommutative Schwinger Effect

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We consider the phenomenon of spontaneous pair production in the presence of an external electric field for noncommutative Yang-Mills theories. Using Maldacena's holographic conjecture, the threshold electric field for pair production is computed from the quark/antiquark potential for noncommutative theories. As an effect of noncommutativity, the threshold electric field is seen to be smaller than its commutative counterpart. We also estimate the correction to the production rate of quark/antiquark pairs to the first order of the noncommutative deformation parameter. Our result bears resemblance with an earlier related work (based on field-theoretic methods).

1. Introduction

Quantum field theory is primarily studied in its perturbative regime. However, there exist quite some novel nonperturbative features of quantum field theories amongst which the Schwinger effect [1] stands its ground (for a recent review, see [2]). The vacuum of quantum electrodynamics is a bath of e^+e^- virtual pairs which gets created and annihilated instantaneously. However, in the presence of an external electric field, the e^+e^- pairs spontaneously become real and their production rate in the weak-coupling weak field approximation is given by [1].

$$\Gamma = \frac{(eE)^3}{(2\pi)^3} e^{-\pi m^2/eE}. \tag{1}$$

This expression holds for weak coupling and weak electric fields only. The exponential suppression hints that the pair production process can be modeled as quantum mechanical tunneling in a certain potential barrier. For the electron-positron pairs to become real, they should gain at least an energy equal to sum of their rest masses ($2m$). However, in the presence of an external electric field, the virtual particles gather an energy of eEx via electromagnetic interactions, x being their separation distance. To understand the situation better, let us assume that the positron is located at distance $-1/2x$ and the electron at distance $1/2x$ from the origin along the direction parallel to the electric field. To become physical particles, the electron has to climb through a potential barrier of height m (same for the positron) by gaining energy from the external electric field. If the virtual pairs are separated by a distance x_* such that $1/2eEx_* = m$ then the electron (and the positron) becomes a real particle as it now has the required rest mass energy. This value $x_* = 2m/eE$ is the width of the potential barrier. Thus, the transmission coefficient is approximately $\exp\left(-x_*\sqrt{2m.m}\right) \sim e^{-4m^2/eE}$. This is exactly the content of the Schwinger formula (1).

Motivated by this analogy, let us look from the perspective of a "virtual" $q\bar{q}$ dipole. When the "virtual" $q\bar{q}$ dipole has a separation x, the total effective potential barrier they encounter can be estimated to be of the form

$$V_{\text{total}}(x) = 2m - \frac{\alpha}{x} - eEx. \tag{2}$$

In this picture, the virtual particles become real by tunneling through the above said potential barrier. The first two terms indicate the self-energy including the Coulombic interaction between $q\bar{q}$ pairs. For small separation, the

Coulomb term dominates the expression and the potential is negative. At large values of x, the effect of electric field takes dominance making the potential negative too. For small electric field, i.e., $E < m^2/e\alpha$, there exist two zero points of the potential profile and the potential is positive in intermediate regimes of separation x. In this case, the particles become real by tunneling through this barrier and the production rate is exponentially suppressed as described by the Schwinger formula. However, for electric fields $E > m^2/e\alpha$, the potential becomes negative all along and ceases to put up a barrier, indicating a catastrophic instability of vacuum where the $q\bar{q}$ are spontaneously produced. The value of electric field for which the potential changes its character is called the "threshold electric field" $\mathscr{E}_T$.

The idea of noncommutative quantum field theories [3, 4] where spacetime position viewed as operators itself ceases to commute was originally proposed to curb the UV divergences appearing in interacting quantum field theories. The idea received revival when some noncommutative field theories were found to be effective low energy limit of open string theories on a Dp brane with a constant NS-NS two-form $B_{\mu\nu}$, the noncommutative feature being a dynamical consequence of quantization [5, 6]. However, the low energy limit of these string theories turns out to be quantum field theories defined on a "noncommutative" spacetime. There exist noncommutative generalizations of Riemannian geometry [7] on which the standard model can be defined, wherein the parameters of the theory are interpreted as geometric invariants. Mathematically, this amounts to abandoning the smooth structure of spacetime in favor of a space equipped with a noncommutative algebra of real-valued functions much like the transition from classical to quantum physics via phase-space methods. The transition from commutative theories to its noncommutative counterpart along the lines stated above is achieved via replacing the ordinary product between functions by the Moyal/star product ($\bigstar$).

$$F(x)G(x) \rightarrow F(x)\bigstar G(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu\partial'_\nu\right)F(x)G\left(x'\right)\Big|_{x=x'}. \tag{3}$$

The above equation implies $[x^\mu, x^\nu]_\bigstar = x^\mu\bigstar x^\nu - x^\nu\bigstar x^\mu = i\theta^{\mu\nu}$, signifying nonvanishing commutation relations between spacetime coordinates (viewed as operators) itself. Noncommutative quantum field theories are generically Lorentz violating (due to the presence of a noncommutative parameter $\theta_{\mu\nu}$), the effects of which are small to be detected in practical experiments with current experimental bound on the noncommutative parameter to be around ($|\theta_{\mu\nu}| \lesssim (10\text{TeV})^{-2}$) by conservative estimates [8].

The Schwinger effect in noncommutative QED has been calculated in [9] where a correction to the pair production rate has been found leading to a decrement in the threshold electric field as a consequence of noncommutativity. However, to carry on the same kind of analysis for strong coupling in general becomes an uphill task and the presence of noncommutativity makes matters worse. The gauge/gravity (holographic) correspondence [10] which links a strongly coupled gauge theory to classical gravity is an important tool in these kinds of scenarios. The Schwinger mechanism has been argued in the holographic setting in [11]. It has also been shown via holographic methods [12] that the Schwinger effect in a large N confining gauge theory admits a "new kind" of critical electric field (apart from the usual threshold value). If the external electric field is less than the confining string tension σ_{str}, the pair production is prohibited as the effective potential barrier remains positive instead of dumping out at large $q\bar{q}$ separation. However, when the electric field is larger than σ_{str}, pair production is allowed as a tunneling process. Thus, at this value (σ_{str}) of electric field, a confinement/deconfinement transition happens.

In this paper, we like to study the holographic Schwinger effect for "quarks" coupled to a large N noncommutative gauge theory in the presence of external $U(1)$ gauge field. The large N noncommutative gauge theory (NCYM) is realized by its relevant holographic dual geometry (to be mentioned later) while the quark is modeled as massive strings extending from the interior to a large but finite position in the holographic direction so that the mass of the quarks is not infinite. We evaluate the interquark potential (both numerically and analytically when possible) for the large N NCYM from the rectangular Wilson loop by calculating the extremal area of a string worldsheet ending along a rectangular contour at the boundary. The effective potential for describing the Schwinger effect is then calculated by introducing an external electric field. We analytically find out the value of the external electric field for which the effective potential puts up a tunneling behavior, i.e., the threshold electric field. We repeat the same calculation for finite temperature NCYM and observe that the thermal contribution to the threshold electric field does not mix up the noncommutative ones. We proceed to find out the decay rate by finding out the on-shell value of the Polyakov action coupled to an external electric field for a string with a circular contour at the boundary. We use perturbations over the known result of circular holographic Wilson loop of ordinary YM and hence find out the correction of the Schwinger decay rate [11] up to the first order of the noncommutative deformation parameter.

This paper is organized as follows. In Section 2, we review the derivation of the Schwinger effect in a superconformal $SU(N)$ gauge theory by relating the same to the expectation value of the circular Wilson loop. Also, in the same section, the basics of noncommutativity in string theory are reviewed for the sake of clarity. Section 3 is devoted to the analysis of effective potential for virtual particles in NCYM plasma, from where the explicit form of the threshold electric field is found out by analytical means. In Section 4, we compute the first order noncommutative correction to the pair production rate of Schwinger particles. We close this paper by conclusions in Section 5.

2. Pair Production and Noncommutativity

2.1. Pair Production in SYM and the Wilson Loop. In its most prominent avatar [10, 13], the holographic duality relates

$\mathcal{N}=4$ super Yang-Mills theory to type IIB string theory in AdS_5S^5. To study the Schwinger effect, one has to account for "massive" matter (corresponding to a probe brane) in fundamental representation and a $U(1)$ gauge field. The way to do so is to break the symmetry group of the problem from $SU(N+1)$ to $SU(N)\,U(1)$ with the Higgs mechanism. Such methods were first introduced in [14]; the following closely follows [15, 16]. For more sophisticated treatment, refer to [17, 18]. The bosonic part of $\mathcal{N}=4$ SYM for the $SU(N+1)$ theory in Euclidean signature reads as

$$S\wedge^{SU(N+1)}=\frac{1}{g_{YM}^2}\int d^4x\left(\frac{1}{4}\widehat{F}_{\mu\nu}^2+\frac{1}{2}\left(D\wedge_\mu\Phi\wedge_i\right)^2-\frac{1}{4}\left[\Phi\wedge_i,\Phi\wedge_j\right]^2\right), \tag{4}$$

where $\widehat{F}_{\mu\nu}$ is the field strength of the $SU(N+1)$ gauge field $\widehat{A}_\mu$. $\widehat{\Phi}_i\,(i=1,\cdots,6)$ collectively denotes six scalars in the adjoint representation of $SU(N+1)$. The gauge group is broken as

$$\widehat{A}_\mu\longrightarrow\begin{pmatrix}A_\mu & \omega_\mu\\ \omega_\mu^\dagger & a_\mu\end{pmatrix},\qquad \widehat{\Phi}_i\longrightarrow\begin{pmatrix}\Phi_i & \omega_i\\ \omega_i^\dagger & m\phi_i\end{pmatrix}. \tag{5}$$

The nondiagonal parts ω_μ and ω_i transform in the fundamental representation of $SU(N)$ and form the so-called W -boson multiplet. The VEV of the $SU(N+1)$ scalar fields is supposed to be of the form

$$\widehat{\Phi}_i=\mathrm{diag}\,(0,\cdots,0,m\phi_i)\,;\qquad \sum_{i=1}^{6}\phi_i^2=1. \tag{6}$$

As a result of the decomposition (5), the $SU(N+1)$ action (4) breaks up into three parts of the following form [14]:

$$S\wedge^{SU(N+1)}\longrightarrow S^{SU(N)}+S^{U(1)}+S_W, \tag{7}$$

where $S^{U(1)}$ is basically the free QED action constructed out of the gauge field a_μ. S_W governs the dynamics of the W bosons and its coupling to the gauge fields. Disregarding the ω_μ's (The ω_μ's start coupling to the ω_i via $\omega_\mu D_\mu\omega_i$. The effect of these couplings to the vacuum energy density of the ω_i is damped by $1/N$ at least. Thus, in the large N limit, the ω_μ are neglected for the present study.) and higher-order terms the W boson action reads

$$S_W=\frac{1}{g_{YM}^2}\int d^4x\left[\left|D_\mu\omega_i\right|^2+\omega_i^\dagger\left(\Phi_j-m\phi_j\right)^2\omega_i-m^2\omega_i^\dagger\phi_i\phi_j\omega_j+\cdots\right], \tag{8}$$

where D_μ is equipped both with the $SU(N)$ gauge field A_μ and also with the $U(1)$ gauge field a_μ, i.e., $D_\mu=\partial_\mu-iA_\mu-i\,a_\mu$. By expanding the action S_W and choosing $\phi_i=(0,0,0,0,0,1)$, the mass term for ω_6 vanishes while those for ω_i, $i=1,\cdots,5$, remain. For the present scenario, the $SU(N)$ gauge field A_μ is a dynamical field and the $U(1)$ gauge field a_μ is a "fixed external" field of the form $a_\mu=a_\mu^{(E)}=-Ex_0\delta_{\mu1}$. The external gauge field contributes to the vacuum energy density via the covariant derivative in S_W as mentioned above. The pair production rate is given by the imaginary part of vacuum energy density [19].

$$\begin{aligned}\Gamma&=-2\,\mathrm{im}\,\ln\int\mathscr{D}A\mathscr{D}\Phi\mathscr{D}\omega\,e^{-S^{SU(N)}-S_W}\\&\approx 5\mathrm{N}\,\mathrm{im}\int\mathscr{D}A\mathscr{D}\Phi\,e^{-S^{SU(N)}}\mathrm{tr}_{SU(N)}\,\widehat{\mathrm{tr}}\,\ln\left(-D_\mu^2+\left(\Phi_i-m\phi_i\right)^2\right).\end{aligned} \tag{9}$$

The factor of N comes from the number of ω_i's. By using Schwinger's parametrization and worldline techniques [20], one can express the pair production rate (9) as a path integral for a particle subject to an appropriate Hamiltonian under boundary conditions, $x(\tau=0)=x(\tau=T)$

$$\Gamma=-5\mathrm{N}\,\mathrm{im}\left\langle\mathrm{tr}_{SU(N)}\mathscr{P}\int_0^\infty\frac{dT}{T}\int\mathscr{D}x(\tau)e^{-\int_0^T d\tau\left[\frac{1}{4}\dot{x}^2+iA_\mu\dot{x}_\mu+ia_\mu^{(E)}\dot{x}_\mu+\left(\Phi_j-m\phi_j\right)^2\right]}\right\rangle. \tag{10}$$

Using saddle point approximations as in [15, 16] and assuming the mass to be heavy, i.e., $m^2>>E$, (10) becomes proportional to the path integral of the particle subjected to the "external" gauge field $a_\mu^{(E)}$ times a phase factor, namely, the $SU(N)$ Wilson loop.

$$\Gamma\sim-5\mathrm{N}\int\mathscr{D}x\,\exp\left(-m\int_0^1 d\tau\sqrt{\dot{x}^2}+i\int_0^1 d\tau\,a_\mu^{(E)}\dot{x}_\mu\right)\langle W[x]\rangle, \tag{11}$$

$$\langle W[x]\rangle=\left\langle\mathrm{tr}_{SU(N)}\mathscr{P}\exp\left(\int_0^1 d\tau\left(iA_\mu\dot{x}_\mu+\Phi_j\phi_j\sqrt{\dot{x}^2}\right)\right)\right\rangle_{SU(N)}. \tag{12}$$

In the above, $\mathscr{P}$ indicates the path ordering of the time parameter τ and the $SU(N)$ nonabelian index is handled with matrix trace, $\mathrm{tr}_{SU(N)}$. Evaluating (11) by the method of steepest descent, the "classical" trajectory becomes a circle. So the production rate is proportional to the expectation value of the circular Wilson loop and can be computed via holographic conjecture in the large N limit.

From expression (11), it naively seems that the pair production rate is nonzero in general implying the vacuum energy (of an $SU(N)$ gauge theory) density to have an imaginary part even in the absence of an external electric field. However, this is not the case. The expression in front of the Wilson loop in (11) can be evaluated by steepest descent methods and leads to a multiplicative factor of $E^2/(2\pi)^3$.

Thus, the pair production rate (and the imaginary part of vacuum energy density) indeed goes to zero when the external electric field vanishes. A detailed calculation of the same is presented in [15, 17].

2.2. Noncommutativity from String Theory. The effective worldsheet action in the presence of $B_{\mu\nu}$ field is given by

$$S = \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2 s\left[\partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} + \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}\right]. \quad (13)$$

The equations of motions along with the boundary conditions when $dB = 0$ are

$$\left(\partial_t^2 - \partial_s^2\right) X^\mu(t,s) = 0, \quad (14)$$

$$\partial_s X^\mu(t,s) + B^\mu_\nu \cdot \partial_t X^\nu(t,s)\big|_{s=0,\pi} = 0. \quad (15)$$

The boundary conditions (15) are neither Neumann nor Dirichlet; one can indeed try to diagonalize (15) to be Neumann-like by redefining the fundamental variables leading to the so-called "open string metric" [6]. However, a more direct attack along the lines of [21] is to solve the equation of motion (14) first and constrain the solution by (15). For $B = B_{23}\, dX^2 \wedge dX^3$, the solution of (15) compatible with (14) is

$$X^2(t,s) = q^2_{(0)} + \left(a^2_{(0)} t + a^3_{(0)} B_{23}\, s\right) + \sum_{n\neq 0}\frac{e^{-\mathrm{int}}}{n}\left(i a^2_{(n)} \cos ns + a^3_{(n)} B_{23} \sin ns\right). \quad (16)$$

A similar solution accompanies $X^3(t,s)$, the forms of which encode the nontrivial boundary conditions. The solutions for the other coordinates are the usual ones [22]. The canonical momentum of the action (14) is by the virtue of mode expansion (16)

$$\begin{aligned}\Pi^2(t,s) &= \frac{1}{2\pi\alpha'}\left(\partial_t X^2(t,s) - B_{23}\,\partial_s X^3(t,s)\right) \\ &= \frac{1}{2\pi\alpha'}\left(a^2_{(0)} + \sum_{n\neq 0} a^2_{(n)}\, e^{-\mathrm{int}} \cos ns\right)\left(1 + (B_{23})^2\right).\end{aligned} \quad (17)$$

The fact that the current scenario leads to noncommutativity was first recognized in [23, 24] as the canonical momentum at the ends of the string becomes functions of the spatial derivatives of the string coordinates as per (15) and (17). From the symplectic 2-form, the canonical commutation relation of the modes is

$$\left[q^2_{(0)}, q^3_{(0)}\right] = i\frac{2\pi\alpha' B_{23}}{1 + (B_{23})^2}, \quad (18)$$

$$\left[q^2_{(0)}, a^2_{(0)}\right] = \left[q^3_{(0)}, a^3_{(0)}\right] = i\frac{2\alpha'}{1 + (B_{23})^2}, \quad (19)$$

$$\left[a^2_{(-n)}, a^2_{(n)}\right] = \left[a^3_{(-n)}, a^3_{(n)}\right] = \frac{2n\alpha'}{1 + (B_{23})^2}. \quad (20)$$

From the mode expansion (16) and the relations (18)–(20), one has

$$\left[X^2(t,s), X^3\left(t,s'\right)\right] = i\frac{2\alpha' B_{23}}{1 + (B_{23})^2}\left[\left(\pi - s - s'\right) - \sum_{n\neq 0}\frac{1}{n}\sin n\left(s + s'\right)\right]. \quad (21)$$

The second term in (21) sums up to zero when $s + s' = 0, 2\pi$. Therefore, the end points of the string become noncommutative. The nontrivial part of the normal ordered Virasoro constraints and the total momentum which accompanies (13) are given by

$$\begin{aligned}L_n &= \frac{1}{4\alpha'} : \sum_m\left[\left(1 + (B_{23})^2\right)\left(a^2_{(n-m)} a^2_{(m)} + a^3_{(n-m)} a^3_{(m)}\right) + \sum_{i,j\neq 2,3}\eta_{ij} a^i_{(n-m)} a^j_{(m)}\right]: \\ P^2_{\text{total}} &= \frac{1}{2\alpha'} a^2_{(0)}\left(1 + (B_{23})^2\right); \\ P^3_{\text{total}} &= \frac{1}{2\alpha'} a^3_{(0)}\left(1 + (B_{23})^2\right).\end{aligned} \quad (22)$$

It is clear from the above that the mass of the particle becomes dependent on the value of the $B_{\mu\nu}$ field. However, the noncommutative field theories constructed out of Moyal product leave the mass of the particle (quadratic part of the Lagrangian) unchanged. Since the string equations of motion and boundary conditions are linear equations, one can redefine the operators to be in terms of which the mass of the theory remains unaltered [25].

$$\begin{aligned}\hat{q}^{2,3}_{(0)} &= \sqrt{1 + (B_{23})^2}\, q^{2,3}_{(0)}; \\ \hat{a}^{2,3}_{(n)} &= \sqrt{1 + (B_{23})^2}\, a^{2,3}_{(n)}.\end{aligned} \quad (23)$$

It terms of which the only nontrivial commutation relation becomes

$$\left[\hat{q}^2_{(0)}, \hat{q}^3_{(0)}\right] = 2\pi i\alpha' B_{23} \equiv i\theta. \quad (24)$$

It has been checked that perturbative string theory in the present backdrop corresponds to noncommutative Yang-Mills at one loop. For further details, see [25] and references there-within.

3. Potential Analysis of the Noncommutative Schwinger Effect

We start with a brief description of the holographic dual of noncommutative Yang-Mills (NCYM) [26, 27]. In the spirit of AdS/CFT correspondence, one looks for supergravity solutions with a nonzero asymptotic value of the B field. Such a solution is the D1-D3 solution which in the string frame looks like

$$ds^2_{str} = \frac{1}{\sqrt{F}}\left[-dx_0^2 + dx_1^2 + H\left(dx_2^2 + dx_3^2\right)\right] + \sqrt{F}\left[dr^2 + r^2 d\Omega_5^2\right];$$
$$F = 1 + \alpha'^2 \frac{R^4}{r^4};$$
$$H = \frac{\sin^2\psi}{F} + \cos^2\psi;$$
$$B = \frac{H \tan\psi}{F} dx_2 \wedge dx_3;$$
$$e^{2\phi} = g_s^2 H. \tag{25}$$

The solution (25) is asymptotically flat and represents $N_{(1)}$ D1 branes dissolved per unit covolume of N D3 branes. The information of the D1 branes is stored in the relation $\tan\psi = N_{(1)}/N$. It can also be seen that the asymptotic value of the B field is $B^{\infty}_{23} = \tan\psi$ while R is related to the other parameters via $R^4 = 4\pi g_s N \cos\psi$.

The proper decoupling limit of the above-stated solution resembles the field-theoretic limit of the noncommutative open string [6, 25], for which the asymptotic value of B_{23} has to be scaled to infinity in a certain way.

$$\begin{aligned} \tan\psi &\longrightarrow \frac{\theta}{\alpha'}; \\ x_{(2,3)} &\longrightarrow \frac{\alpha'}{\theta} x_{(2,3)}; \\ r &\longrightarrow \alpha' R^2 u; \\ g_s &\longrightarrow \frac{\alpha'}{\theta}\hat{g}; \\ \alpha' &\longrightarrow 0. \end{aligned} \tag{26}$$

With the above scaling and keeping $x_{(2,3)}, u, \hat{g}, \theta$ fixed the resulting metric and field configurations are given by

$$ds^2_{str} = \alpha'\sqrt{\lambda}u^2\left[-dx_0^2 + dx_1^2 + h\left(dx_2^2 + dx_3^2\right)\right] + \alpha'\sqrt{\lambda}\frac{du^2}{u^2} + \alpha'\sqrt{\lambda}d\Omega_5^2;$$
$$h = \frac{1}{1 + \lambda\theta^2 u^4};$$
$$B_{23} = \frac{\alpha'\lambda\theta u^4}{1 + \lambda\theta^2 u^4};$$
$$e^{2\phi} = \hat{g}h;$$
$$\lambda \equiv R^4 = 4\pi\hat{g}N. \tag{27}$$

The above is the holographic dual to NCYM with gauge group $SU(N)$ and Yang-Mills coupling constant $g_{\text{NCYM}} = \sqrt{4\pi\hat{g}}$ which captures the dynamics of NCYM. As the holographic direction u tends to infinity, the first two directions x_0, x_1 scale as u^2 and the x_2, x_3 directions develop a $1/u^2$ dependence in the metric. Due to the noncommutativity, the symmetry group of the theory becomes $SO(1,1) \otimes SO(2)$ as can be clearly seen from the isometry of (27) [28]. The gravity dual to NCYM at finite temperature T is found from the near horizon limit of the black D1-D3 solution and reads

$$ds^2_{str} = \alpha'\sqrt{\lambda}u^2\left[-\left(1 - \frac{\pi^4 T^4}{u^4}\right)dx_0^2 + dx_1^2 + \frac{1}{1 + \lambda\theta^2 u^4}\left(dx_2^2 + dx_3^2\right)\right] + \frac{\alpha'\sqrt{\lambda}}{\left(1 - \left(\pi^4 T^4/u^4\right)\right)}\frac{du^2}{u^2},$$
$$B_{23} = \frac{\alpha'\lambda\theta u^4}{1 + \lambda\theta^2 u^4}. \tag{28}$$

The most rigorous approach to study the Schwinger effect in the holographic setting is to find the expectation value of the circular Wilson loop and relate it to the pair production rate as in [11]. However, one may think of the vacuum to be made of $q\bar{q}$ pairs bound under an attractive potential and study how an external electric field modifies this potential. This is the essence of potential analysis which was first put forward in [29]. To compute the interquark potential, one needs to look at the expectation value of the rectangular Wilson loop when the loop contour is regarded as the trajectory of particles under consideration in the $x_0 - x_3$ plane where x_3 is the direction of $q\bar{q}$ orientation. As pointed out in [11, 30], one places a probe D brane at a finite position instead of the boundary to get a W boson of finite mass. A string with Dirichlet conditions at both ends has the following canonical Hamiltonian [22], where the first term indicates the potential energy of the stretched string and is the analogue of mass created due to symmetry breaking.

$$H = \frac{\left(q_a^\mu - q_b^\mu\right)^2}{4\pi\alpha'} + \sum_{n\neq 0}\alpha_{(-n)}\alpha_{(n)}. \tag{29}$$

As per the holographic procedure, the VeV of the Wilson loop of a gauge theory is given by the partition function of a fundamental string in the background of the holographic dual with the ends of the string anchored on the probe D brane along the contour of the Wilson loop $\mathscr{C}$ (12), i.e.,

$$\langle W[\mathscr{C}]\rangle = \frac{1}{\text{Vol}}\int_{\partial X = \mathscr{C}} \mathscr{D}X\mathscr{D}h_{ab}e^{-S[X,h]}, \tag{30}$$

where $S[X,h]$ indicates the action of the fundamental string (The boundary conditions are given by the trajectory of the string at the probe brane, $\partial X = \mathscr{C}$. Stated more explicitly, $\mathscr{D}X = \mathscr{D}\xi$ where $X^\mu(s,t) = c^\mu(t) + \xi^\mu(t,s)$. The functions $\xi^\mu(t,s)$ vanishes at the boundary which is given by a specific value $s = s_0$, $c^\mu(t)$ being the parametric representation of the contour $\mathscr{C}$ of the Wilson loop.) which has been Wick rotated to Euclidean signature, as is usually done in string theory [22]. In the classical limit which is realized when the string length α' is small (or the 't Hooft coupling is big), the above expression is dominated by the on shell value of the Polyakov/Nambu-Goto action. Thus, the prescription of computing Wilson loops is reduced to computing the area of the world sheet of the fundamental string which ends on the specified profile at the probe D brane [14, 31, 32], situated at a finite radial position in the dual geometry for the present case.

3.1. Potential Analysis at Zero Temperature. In this section, we will study the properties of the modified potential of NCYM in the presence on a constant external electric field for quark/antiquark pairs along one of the noncommutative directions (x_3). The appropriate holographic dual as pointed above is given by (27). To calculate the extremal area of the string configuration ending of the probe brane, we take the string world sheet to be parametrized by $s^a \equiv (s, t)$. The Nambu-Goto action reads

$$\mathcal{S}_{NG} = \frac{1}{2\pi\alpha'} \int dt ds \sqrt{\det G^{(\mathrm{in})}_{ab}}, \qquad G^{(\mathrm{in})}_{ab} \equiv G_{\mu\nu} \frac{\partial x^\mu}{\partial s^a} \frac{\partial x^\nu}{\partial s^b}. \tag{31}$$

In the above, $G^{(\mathrm{in})}_{ab}$ is the induced metric on the world-sheet and $G_{\mu\nu}$ is the metric of the target spacetime/holographic dual (27) and (28). The above action exhibits two diffeomorphism symmetries with the help of which one can set two of the embedding functions to arbitrary values provided that the resulting profile matches with the contour of our choosing on the probe brane. One usually chooses the so-called static gauge for which the profile reads (Strictly speaking, the embedding function which extremizes the Nambu-Goto function may not respect such a gauge choice globally and may lead us to a local minimum of the Nambu-Goto action. One can hope the results found will converge to the true value if perturbations are added.)

$$\begin{gathered} x^0(s,t) = t; \\ x^3(s,t) = s; \\ u(s,t) = u(s); \\ x^{1,2} = 0; \\ \Theta^i(s,t) = \text{constant}. \end{gathered} \tag{32}$$

In the above, the extremization of the Nambu-Goto action is given by the functional form of $u(s)$. The Θ^i are the coordinates of S^5. For present purposes, $x_3 \equiv s$ is assumed to range between $[-L, L]$, when $2L$ indicates the interquark separation on the probe brane with the boundary condition $u(\pm L) = u_B$ where u_B indicates the position of the probe brane along the holographic direction. Again, the temporal direction is assumed to range from $[-\mathcal{T}, \mathcal{T}]$ with the further assumption that $\mathcal{T} \gg L$, meaning time scale of the problem (within which the quarks/antiquark pairs remain separated) to be much larger than the length scale. This is because the rectangular Wilson loop gives the interquark potential when one assumes the interaction between dipole is adiabatically switched on and off as illustrated in Figure 1 (Intuitively, one thinks, two quarks to be separated at distant past and then reunite at distant future. Thus, the worldline becomes a rectangular closed contour whose area can be identified with the interquark potential $W[\mathscr{C}] \sim e^{-\mathcal{T} U(L)}$.).

Before proceeding further, let us address the issue of the B field. For noncommutative gauge theories, the B field is excited (27) and (28) and is present in the string action via the Wess-Zumino term $\int dt ds B_{\mu\nu} \partial_t x_\mu \partial_s x_\nu$ along with the usual Polyakov/Nambu-Goto part. However, the gauge choice given above (32) cancels the contribution of the Wess-Zumino part of the action. It is possible to consider the $q\bar{q}$ pairs at a velocity in the x_2 direction and take into account the contribution of the $B_{\mu\nu}$ term as in [33]. However, in the present case where the virtual particles in vacuum are modeled as $q\bar{q}$ dipoles, such a configuration seems hardly sensible.

As per the above gauge choice, the induced metric/line element on the world sheet reads

$$G^{(\mathrm{in})}_{ab} ds^a ds^b = -\alpha' \sqrt{\lambda} u^2 dt^2 + \frac{\alpha' \sqrt{\lambda}}{u^2} \left[\left(\frac{du}{ds} \right)^2 + \frac{u^4}{1 + \lambda\theta^2 u^4} \right] ds^2. \tag{33}$$

Using the above form of the induced metric in the action (31), we get

$$\mathcal{S}_{NG} = \frac{\mathcal{T}}{2\pi\alpha'} \int_{-L}^{L} ds \sqrt{\left(\frac{du}{ds} \right)^2 + \frac{u^4}{1 + \lambda\theta^2 u^4}}. \tag{34}$$

Extremization of the above Lagrangian is equivalent to solving the Euler-Lagrange equation with the effective Lagrangian $\mathscr{L}_{NG} = \sqrt{(du/ds)^2 + (u^4/(1 + \lambda\theta^2 u^4))}$ when $u = u(s)$ along with the boundary condition $u(s \equiv x_3 = \pm L) = u_B$ as the contour profile is already taken into account by the gauge choice. One can indeed solve the relevant problem and find the explicit form of $u(s \equiv x_3)$. However, since the Lagrangian in (34) does not explicitly depend on the parameter s (In principle, one considers a profile where $u = u(s, t)$ as the two diffeomorphism symmetries are exhausted by (32). However, such a choice would reflect nonadiabatic interactions, i.e., the interpretation of Wilson loops of measuring the interaction potential of $q\bar{q}$ pairs at rest as in Figure 1 would be invalid.), by Noether's theorem, there exists a conserved quantity (Q) for the solution.

$$Q \equiv -\frac{du}{ds} \frac{\partial \mathscr{L}_{NG}}{\partial (du/ds)} + \mathscr{L}_{NG} = \frac{u^4 h(u)}{\sqrt{(du/ds)^2 + u^4 h(u)}}; \qquad h(u) = \frac{1}{1 + \lambda\theta^2 u^4}. \tag{35}$$

As indicated in [32], the fundamental string is assumed to be carrying charges at its two ends and is otherwise symmetric about its origin. Thus, one can choose $u(s)$ to be an even function of $s \equiv x_3$. In the present case, this means the x_3 direction of NCYM is symmetric around its origin which

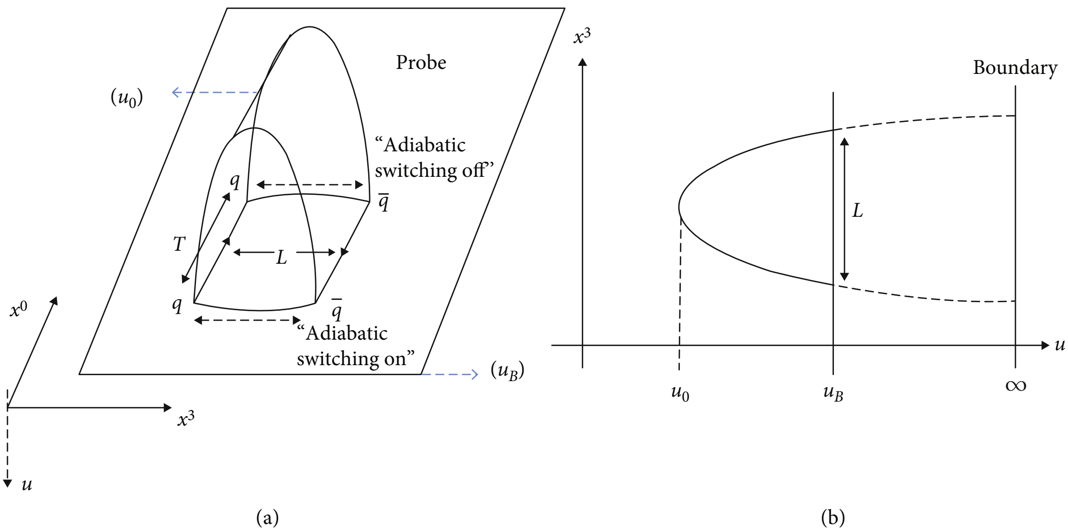


Figure 1: This figure illustrates the setup used. The probe brane is placed at a finite position (u_B) on the holographic direction as in (b). On the probe brane, the placement of the Wilson loop is shown in (a), arrows indicating the contour of the loop (not the propagation of the string). For adiabatic interactions, one can neglect the effects of the dotted lines and the string profile becomes static.

holds true in spite of its noncommutative nature. From the form of (35), the solution of du/ds involves both positive and negative signs. Thus, there exist a value of the parameter $s = s_0$ for which $du/ds(s_0) = 0$. This is the turning point of the string profile as indicated in Figure 1. Simplifying (35) and introducing a rescaled holographic coordinate $y = u/u_0$, one obtains the following differential equation:

$$\frac{d}{ds}\left(\frac{u}{u_0}\right) \equiv \frac{dy}{dx_3} = \frac{u_0 y^2 \sqrt{y^4 - 1}}{1 + \lambda\theta^2 u_0^4 y^4}. \quad (36)$$

In the above, u_0 indicates the value of $u(s)$ at $s = s_0$ and the gauge choice $x_3(s,t) = s$ has been used. The above equation is obtained from evaluating the l.h.s of (35) at the turning point. From the equation obtained, one can estimate the separation length ($2L$) of the $q\bar{q}$ dipole by integration both sides

$$L \equiv \int_0^L dx_3 = \int_1^{u_B/u_0} \frac{dy}{u_0 y^2 \sqrt{y^4 - 1}} + \int_1^{u_B/u_0} \frac{\lambda\theta^2 u_0^3 y^2}{\sqrt{y^4 - 1}}\, dy. \quad (37)$$

It is worthwhile to point out that if one tries to take $u_B \longrightarrow \infty$ in (37), the dipole length diverges due to the second integral (which is absent in the commutative counterpart where $\theta = 0$). However, unlike the generic quark/antiquark potential calculation where a divergence is attributed to the self-energy of infinitely massive quarks, the present situation cannot be remedied by such arguments. The fact that the holographic dual of a NCYM does not live at radial infinity has been reported in [26] where it has been shown a slight perturbation on the string profile at infinity destabilizes it completely. An alternative has been advocated in [33] where the string profile is allowed to have a velocity in the x_2 and the $B_{\mu\nu}$ term in the string action contributes to the interquark length unlike the present case. It can be shown that for a certain velocity of the $q\bar{q}$ pair in the transverse direction, the dipole can be consistently taken to radial infinity. As indicated before, we avoid such a configuration for the present case.

The mass of the fundamental matter ($q\bar{q}$ pairs) is given by the self-energy of a stretched string from the probe to the interior [34]. For determining the same, the relevant gauge is $x_0 = t$, $u = s$, $x_3 = \text{constant}$. Thus, the mass is given by

$$m = \frac{1}{2\pi\alpha'}\int_0^{u_B} du \sqrt{\alpha'\sqrt{\lambda}u^2 \cdot \frac{\alpha'\sqrt{\lambda}}{u^2}} = \frac{\sqrt{\lambda}}{2\pi} u_B. \quad (38)$$

The dipole separation length of the test particles (37) can be analytically integrated, and in terms of the parameter $a = u_0/u_B$, one has

$$L = \frac{\sqrt{\lambda}}{2\pi m}\left[\frac{\sqrt{\pi}\,\Gamma(3/4)}{a\,\Gamma(1/4)} - \frac{a^2}{3}\,{}_2F_1\left(\frac{1}{2},\frac{3}{4},\frac{7}{4},a^4\right)\right] + \frac{8\pi^3 m^3\theta^2}{\sqrt{\lambda}}a^3 \cdot \left[-\frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} + a\cdot{}_2F_1\left(-\frac{1}{4},\frac{1}{2},\frac{3}{4},a^4\right)\right]. \quad (39)$$

The sum of the potential and the static energy of the $q\bar{q}$ pairs is given by the (time averaged) on-shell value of the Nambu-Goto action which by the virtue of (34) and (37) is

$$\begin{aligned} U_{CP+SE} &= \frac{\sqrt{\lambda}}{2\pi} a u_B \int_1^{1/a} dy\, \frac{y^2}{\sqrt{y^4-1}}\sqrt{1+\lambda\theta^2 a^4 u_B^4} \\ &= m\sqrt{1+\frac{16\pi^4 m^4\theta^2 a^4}{\lambda}}\left[-a\frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} + {}_2F_1\left(-\frac{1}{4},\frac{1}{2},\frac{3}{4},a^4\right)\right]. \end{aligned} \quad (40)$$

From the above, one can look at the limit when $L \longrightarrow \infty$

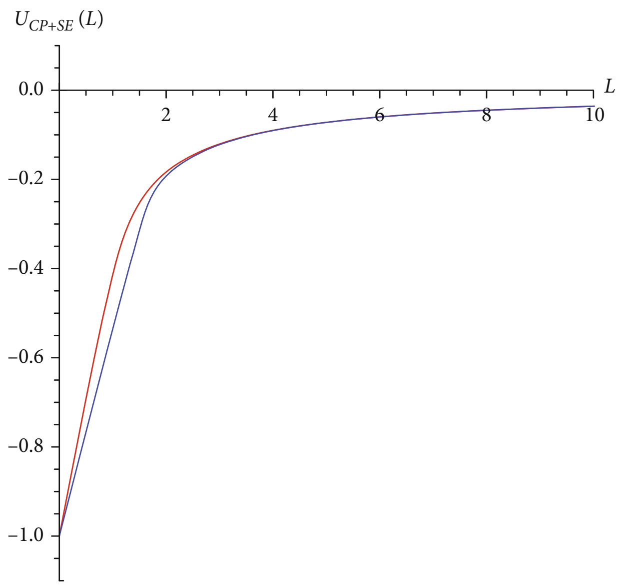


FIGURE 2: This is a parametric plot of $L(a)$ v/s a. The values chosen are $m = 1$, $\lambda = 4\pi^2$. The red and blue lines are plots for $\theta = 0.2$ and $\theta = 0.3$, respectively; the cyan line stands for $\theta = 0.75$. Note that for the latter case, the plot becomes degenerate of intermediate values of the parameter a, though the predictions for extreme values of L remain true. We will stick to the first two values for our numerical computations.

which is the same as taking the limit $a \longrightarrow 0$. This fact is evident from the above expression, see Figure 2. This is the situation when the string stretches to the interior, i.e., $u_0 \longrightarrow 0$. Using the relation ${}_2F_1(-1/4, 1/2, 3/4, 0) = 1$, the leading dependence of the interquark length (39) is given by $L = 1/2m\sqrt{(\lambda/\pi)}((\Gamma(3/4))/(a\,\Gamma(1/4)))$, and the interquark potential (40) becomes

$$U_{CP+SE}(a \longrightarrow 0) \approx m - am\frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} = m - \frac{\sqrt{\lambda}}{2}\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^2\frac{1}{L}. \tag{41}$$

Thus, the usual Coulomb law is recovered at large distances. However, for arbitrary separation, there is a rapid modification from the Coulomb law. This can be attributed due to two reasons. Firstly, as is evident from (39) and (40) for arbitrary values of the parameter a, the noncommutative effects creep in which breaks the conformal symmetry and hence Coulombic dependence. Secondly, as found in [29], even for a commutative theory, the potential profile is altered from and is finite even at short distances. This is because in the presence of a mass term, the theory is not conformal anymore as can be understood from the presence of (8) which is coupled to the usual $SU(N)$ action. To get a better view of the same, we look at a 1 limit of (40) and (39). It is evident from the integrals that both of them vanishes in the above-said limit. Looking at the limiting values, one has

$$U_{CP+SE}(a \longrightarrow 1^-) \approx \left[\frac{m/\lambda}{\sqrt{1 + (16\pi^4 m^4\theta^2 a^4/\lambda)}}\left(-\frac{16\pi^4 m^4\theta^2 a^4 + \lambda}{a\sqrt{1-a^4}} - \frac{\sqrt{\pi}(48\pi^4 m^4\theta^2 a^4 + \lambda)\Gamma(3/4)}{\Gamma(1/4)} + \frac{(48\pi^4 m^4\theta^2 a^4 + \lambda)\,{}_2F_1(-1/4, 1/2, 3/4, a^4)}{a}\right)\right]_{a\longrightarrow 1^-} \cdot (1-a). \tag{42}$$

In the above, the $\mathcal{O}(1-a)^2$ terms have been neglected. Similarly for the interquark distance, one has

$$L(a \longrightarrow 1^-) \approx (1-a)\cdot\frac{\sqrt{\lambda}}{2\pi m}\left[-\frac{a}{\sqrt{1-a^4}} + \frac{a}{3}{}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, a^4\right) - \frac{\sqrt{\pi}}{a^2}\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]_{a\longrightarrow 1^-} + (1-a)\cdot\left[-\frac{8\pi^3 m^3\theta^2 a^3}{\sqrt{\lambda}\sqrt{1-a^4}} + \frac{16\pi^3 m^3\theta^2 a^3}{\sqrt{\lambda}}\,{}_2F_1\left(\frac{-1}{4}, \frac{1}{2}, \frac{3}{4}, a^4\right)\right]_{a\longrightarrow 1^-}. \tag{43}$$

Comparing (42) and (43) and using the identity ${}_2F_1(a, b, c, 1) = (\Gamma(c)\Gamma(c-a-b))/(\Gamma(c-a)\Gamma(c-b))$, it can be easily seen that for short interquark separation,

$$U_{CP+SE}(a \longrightarrow 1^-) \approx \frac{2\pi m^2}{\sqrt{\lambda}}\frac{1}{\sqrt{1 + (16\pi^4 m^4\theta^2/\lambda)}}L(a \longrightarrow 1^-). \tag{44}$$

Thus, increment in the value of noncommutative parameter θ results into decrement of the slope of the potential curve at small separation. This is because of repulsive forces due to noncommutativity, signaling the force needed to detach the noncommutative $q\bar{q}$ pair should be smaller than its commutative counterpart.

Till now, we have calculated the interquark potential. However, in the presence of an external electric field, the charged $q\bar{q}$ pairs develop an electrostatic potential of their own. The total potential is given by the sum of the two. Defining the effective potential to be

$$U_{\text{effective}}(L) = U_{CP+SE}(L) - E\cdot L. \tag{45}$$

It can be guessed from (44) that in the presence of an external electric field of strength,

$$\mathscr{E}_T = \frac{2\pi m^2}{\sqrt{\lambda}}\frac{1}{\sqrt{1 + (16\pi^4 m^4\theta^2/\lambda)}}, \tag{46}$$

the $q\bar{q}$ pair overcomes the linear barrier of the potential profile in Figure 3. However, it is still to be seen whether the above-mentioned electric field is enough to get out of the tunneling phase for all values of interquark separation L. One has from (37) and (40),

$$U_{\text{effective}}(L(u_0)) = (1-r)\mathscr{E}_T L(u_0) + G(L(u_0)),$$

$$G(L(u_0)) = \frac{\sqrt{\lambda}}{2\pi}\int_{u_0}^{u_B} du \left[\frac{u^2\sqrt{1+\lambda\theta^2 u_0^4}}{\sqrt{u^4-u_0^4}} - \frac{u_B^2 u_0^2}{\sqrt{1+\lambda\theta^2 u_B^4}}\frac{1+\lambda\theta^2 u^4}{u^2\sqrt{u^4-u_0^4}}\right],$$

$$L(u_0) = \int_{u_0}^{u_B} du \frac{u_0^2}{u^2}\frac{(1+\lambda\theta^2 u^4)}{\sqrt{u^4-u_0^4}}. \tag{47}$$

In the above, we have reinstated the turning point u_0 and have introduced the ratio $r = E/\mathscr{E}_T$ for simplicity where $\mathscr{E}_T$ the threshold electric field is given by (46). It is apparent from (47), at the value $r = 1$ (applied field being of threshold value), the first term vanishes. Thus, the potential profile is governed fully by the second term $G(L(u_0))$ and will cease to put up a tunneling barrier if the function is monotonically decreasing and is vanishing at the origin. At $L = 0$ which is realized if $u_0 = u_B$, it is evident from (47) that $G(L=0) = 0$. Indeed, this is the case for $U_{\text{effective}}(L=0)$ too.

$$\frac{d}{dL}U_{\text{effective}}(L) = (1-r)\mathscr{E}_T + \frac{d}{dL}G(L) = (1-r)\mathscr{E}_T + \frac{du_0}{dL}\frac{dG(u_0)}{du_0}. \tag{48}$$

Moreover, one can show from (47) that

$$\frac{dL(u_0)}{du_0} = -\frac{(1+\lambda\theta^2 u_0^4)}{\sqrt{(u_0+\varepsilon)^4-u_0^4}} + \int_{u_0}^{u_B} du \cdot \left[\frac{2u_0(1+\lambda\theta^2u^4)}{u^2\sqrt{u^4-u_0^4}} + \frac{2u_0^5(1+\lambda\theta^2u^4)}{u^2\left(\sqrt{u^4-u_0^4}\right)^3}\right]$$
$$= \left[-\frac{(1+\lambda\theta^2 u_0^4)}{\sqrt{(u_0+\varepsilon)^4-u_0^4}} + 2\int_{u_0}^{u_B} du \frac{(1+\lambda\theta^2 u^4)}{\left(\sqrt{u^4-u_0^4}\right)^3}u_0u^2\right]. \tag{49}$$

The first term comes when the differential operator acts on the lower limit of integration in (47). A regulator ε (whose physical meaning is rather vague) has been put in the expression as the first term is actually divergent. By a similar procedure, one has

$$\frac{dG(u_0)}{du_0} = \frac{\sqrt{\lambda}}{2\pi}\left[-\frac{(1+\lambda\theta^2 u_0^4)}{\sqrt{(u_0+\varepsilon)^4-u_0^4}} + 2\int_{u_0}^{u_B} u_0u^2\frac{(1+\lambda\theta^2u^4)}{\left(\sqrt{u^4-u_0^4}\right)^3}\right] \times \left(\frac{u_0^2\sqrt{1+\lambda\theta^2u_B^4} - u_B^2\sqrt{1+\lambda\theta^2u_0^4}}{\sqrt{(1+\lambda\theta^2u_B^4)(1+\lambda\theta^2u_0^4)}}\right). \tag{50}$$

Though the above two terms are actually divergent, one can see from (49) and (50) that their ratio is not.

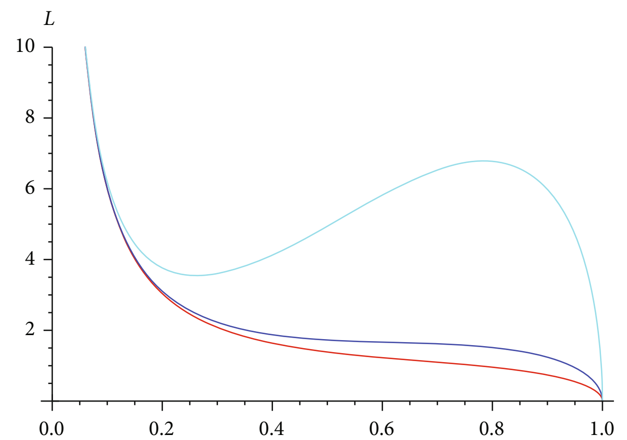


FIGURE 3: This is a parametric plot of U_{CP+SE} v/s L. The plot indicates the profile of the potential (with static energy subtracted) to be Coulombic at large distances. At small distance, the profile exhibits a linear dependence. However, noncommutativity has a tendency to increase the linear effect. The values used are $m = 1$ and $\lambda = 4\pi^2$. The red line stands for the value $\theta = 0.2$ while the blue line indicates $\theta = 0.3$.

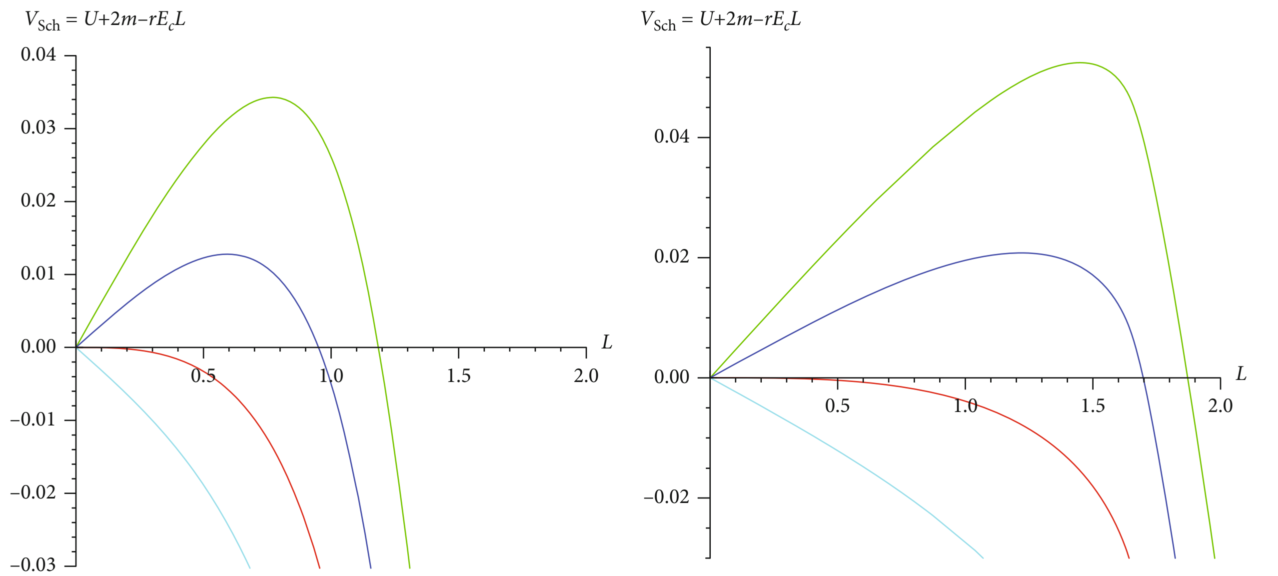
$$G'(L) \equiv \frac{dG(L)}{dL} = \frac{\sqrt{\lambda}}{2\pi}\left(\frac{u_0^2}{\sqrt{1+\lambda\theta^2u_0^4}} - \frac{u_B^2}{\sqrt{1+\lambda\theta^2u_B^4}}\right). \tag{51}$$

It is easily seen that $G(L(u_0))$ is a monotonically decreasing function w.r.t $L(u_0)$, i.e., for $u_B \geq u_0$. From the above, it is clear that the net potential/force due to the applied field has two components as follows:

(a) $(1-r)\mathscr{E}_T L(u_0)$: this is the part which creates the potential barrier for $r < 1$, i.e., the attractive force between the $q\bar{q}$ pairs in an external electric field. At $r = 1$, this part ceases to contribute, and for $r > 1$, the force corresponding to this part becomes repulsive

(b) $G(L(u_0))$: this part contributes to bringing down the potential barrier. As can be seen from (51), the associated force due to this is repulsive for all values of $L(u_0)$ except at the origin where it vanishes. Thus, at threshold point ($r = 1$) where the first component (part a) becomes irrelevant, the slope of the net potential for all nonzero values L is negative as can be seen in Figure 4 confirming the prediction of (46)

It is also interesting to see whether the effective potential admits a confining phase where the tunneling behavior is totally absent. This amounts to showing the existence of an intermediate value of u_0 for $r < 1$ where the total potential as in (45) and (47) vanishes. Alternatively, one can check the values of the slope of the potential at extreme points and look for a saddle point of the same. It is easy to see that at $L = 0/u_0 = u_B$, the slope (48) is given by $(1-r)\mathscr{E}_T$ which



(a) $\theta = 0.2$

(b) $\theta = 0.3$

Figure 4: The plot indicates the effective potential (in the presence of external electric field) v/s the interquark separation. The values used are $m = 1$ and $\lambda = 4\pi^2$. The green line indicates $r = 0.9$, blue line for $r = 0.95$. The parameter r is the ratio of the applied field to its threshold value. Note that the maximal height of the potential barrier decreases as noncommutativity increases. The red line which exhibits the threshold behavior stands for $r = 1.0$, and cyan for $r = 1.05$ shows catastrophic decay of vacuum.

is positive for $r < 1$. However, at $u_0 = 0/L \longrightarrow \infty$, the slope of the potential is given by

$$\frac{d}{dL} U_{\text{effective}}(L \longrightarrow \infty) = -r \frac{\sqrt{\lambda}}{2\pi} \frac{u_B^2}{\sqrt{1+\lambda\theta^2 u_B^4}}. \quad (52)$$

Thus, we see that the slope of the potential curve is negative (force between $q\bar{q}$ pairs is repulsive) at large distances for all values of the applied electric field unlike the situations in [12, 35], indicating the present case of not being confining. It is also clear from (48) and (51) that the maximal potential barrier is encountered when

$$a^4 \equiv \left(\frac{u_0}{u_B}\right)^4 = \frac{r^2}{1 + \left(16\pi^4 m^4 \theta^2/\lambda\right)\left(1-r^2\right)}. \quad (53)$$

This is the point when the effective $q\bar{q}$ potential becomes repulsive rather than being attractive.

3.2. Potential Analysis at Finite Temperature. The finite temperature case closely resembles the above calculation. For the present situation, the gravity dual is given by (28). The thermal mass of fundamental $q\bar{q}$ pairs is given by

$$\begin{aligned} m(T) &= \frac{1}{2\pi\alpha'} \int_{\pi T}^{u_B} du \sqrt{\alpha' \sqrt{\lambda} u^2 \left(1 - \frac{\pi^4 T^4}{u^4}\right) \cdot \frac{\alpha' \sqrt{\lambda}}{u^2 \left(1 - \left(\pi^4 T^4/u^4\right)\right)}} \\ &= \frac{\sqrt{\lambda}}{2\pi}\left(u_B - \pi T\right) = m(T=0) - \frac{\sqrt{\lambda}}{2} T. \end{aligned} \quad (54)$$

Very much like the above analysis, the Nambu-Goto action in static gauge reduces to

$$S = \mathcal{T} \frac{\sqrt{\lambda}}{2\pi} \int ds \sqrt{\left(\frac{du}{ds}\right)^2 + \frac{u^4 - \pi^4 T^4}{1+\lambda\theta^2 u^4}}. \quad (55)$$

Quite similar to the previous case, the conserved quality arising from the Lagrangian (55) is

$$\frac{1}{1+\lambda\theta^2 u^4} \cdot \frac{u^4 - \pi^4 T^4}{\sqrt{(du/ds)^2 + \left(\left(u^4 - \pi^4 T^4\right)/\left(1+\lambda\theta^2 u^4\right)\right)}} = \text{conserved}. \quad (56)$$

Demanding the profile admits a zero slope at the turning point u_0 (the turning point should be greater than the horizon radius, i.e., $u_0 \geq \pi T$), we have

$$\frac{du}{ds} = \frac{\sqrt{\left(u^4 - \pi^4 T^4\right)\left(u^4 - u_0^4\right)\left(1+\lambda\theta^2\pi^4 T^4\right)}}{\sqrt{\left(u_0^4 - \pi^4 T^4\right)\left(1+\lambda\theta^2 u^4\right)}}. \quad (57)$$

From the above equation, the separation length between test particles can be integrated out to be

$$L_T(a) = \frac{1}{a}\frac{\sqrt{\lambda}}{2\pi m} \int_1^{1/a} dy \frac{\sqrt{1 - \left(\lambda^2 T^4/16 m^4 a^4\right)\left(1 + \left(16\pi^4 m^4 \theta^2/\lambda\right) y^4 a^4\right)}}{\sqrt{\left(y^4 - 1\right)\left(y^4 - \left(\lambda^2 T^4/16 m^4 a^4\right)\right)\left(1+\lambda\theta^2\pi^4 T^4\right)}}. \quad (58)$$

The above equation is written in terms of redefined variables: $y = u/u_0$; $a = u_0/u_B$; the parameter m is the mass at zero temperature. The interquark potential at finite temperature for noncommutative theories is obtained from (55) and (58) to be

$$U_T(L_T(a)) = ma\int_1^{\frac{1}{a}} dy \sqrt{\frac{y^4 - (\lambda^2 T^4/16m^4a^4)}{y^4-1}}\sqrt{\frac{1+(16\pi^4 m^4\theta^2 a^4/\lambda)}{1+\lambda\theta^2\pi^4 T^4}}. \tag{59}$$

It is not possible to integrate the above two equations analytically. The separation however due to the presence of a finite temperature, the interquark potential ceases to be Coulombic even at large interquark separation which can be explicitly checked by computing (59) for small temperature using binomial approximation. This phenomenon can be attributed to the breakdown of conformal symmetry (for the commutative case) in finite temperature.

In the presence of an external electric field E, the effective potential experienced by the $q\bar{q}$ pairs gets modified to

$$\begin{aligned} U_{T,\text{eff}}(L_T(a)) &= U_T(L_T(a)) - E\cdot L_T = (1-R)\,\mathscr{E}_{th}\cdot L_T(a) \\ &\quad + \frac{\sqrt{\lambda}}{2\pi}H(L_T(a))\,;\ R = \frac{E}{\mathscr{E}_{th}}. \end{aligned}$$

$$\begin{aligned} H(a) &= \int_1^{1/a} dy \frac{1}{\sqrt{(y^4-1)(1+\lambda\theta^2\pi^4T^4)}} \\ &\quad\cdot\left[\frac{2\pi m}{\sqrt{\lambda}}a\sqrt{\left(y^4 - \frac{\lambda^2T^4}{16m^4a^4}\right)\left(1+\frac{16\pi^4m^4\theta^2a^4}{\lambda}\right)}\right. \\ &\quad\left. - \frac{\mathscr{E}_{th}}{am}\cdot\frac{\sqrt{1-(\lambda^2T^4/16m^4a^4)}\left(1+(16\pi^4m^4\theta^2a^4/\lambda)y^4\right)}{\sqrt{y^4-(\lambda^2T^4/16m^4a^4)}}\right]. \end{aligned} \tag{60}$$

In the first step, we have added and subtracted the term "$\mathscr{E}_{th}\cdot L_T$" where $\mathscr{E}_{th}$ the threshold electric field at finite temperature is to be found out. The slope of the potential profile (60) for fixed values of the physical parameters is given by

$$\frac{dU_{T,\text{eff}}(a)}{dL_T} = (1-R)\mathscr{E}_{th} + \frac{\sqrt{\lambda}}{2\pi}\cdot\frac{dH(a)}{da}\cdot\left(\frac{dL_T(a)}{da}\right)^{-1}. \tag{61}$$

At the threshold point where $R=1$, the first term vanishes and one is left with the second term alone which itself consists of the threshold value (60). However, at the threshold point, the slope the potential should be negative for all allowed values of the parameter a (and henceforth the separation L_T). An explicit calculation leads to

$$\begin{aligned} \frac{dL_T(a)}{da} &= \frac{\sqrt{\lambda}}{2\pi m}\frac{1}{a^2}\frac{1}{\sqrt{1+\lambda\theta^2\pi^4T^4}} \\ &\quad\cdot\left[\frac{2}{\sqrt{1-(\lambda^2T^4/16m^4a^4)}}\right. \\ &\quad\cdot\int_1^{1/a} dy\frac{\sqrt{y^4-(\lambda^2T^4/16m^4a^4)}\left(1+(16\pi^4m^4\theta^2a^4/\lambda)y^4\right)}{\sqrt{(y^4-1)^3}} \\ &\quad\left.-\left(\frac{1+(16\pi^4m^4\theta^2a^4/\lambda)}{\sqrt{(1+\varepsilon)^4-1}}\right)\right]. \end{aligned} \tag{62}$$

Similarly (after quite some algebraic manipulations), one finds

$$\frac{dH(a)}{da} = \left(\frac{4\pi^2m^2}{\lambda}a^2\sqrt{\frac{1-(\lambda^2T^4/16m^4a^4)}{1+(16\pi^4m^4\theta^2/\lambda)a^4}} - \frac{2\pi}{\sqrt{\lambda}}\mathscr{E}_{th}\right)\frac{dL_T(a)}{da}, \tag{63}$$

$$\Longrightarrow \frac{dU_{T,\text{eff}}(a)}{dL_T} = (1-R)\mathscr{E}_{th} + \left(\frac{2\pi m^2}{\sqrt{\lambda}}a^2\sqrt{\frac{1-(\lambda^2T^4/16m^4a^4)}{1+(16\pi^4m^4\theta^2/\lambda)a^4}} - \mathscr{E}_{th}\right). \tag{64}$$

It is clear from (58) that as the parameter $a \longrightarrow 1$, the interquark separation $L_T \longrightarrow 0$. Moreover, at $a=1$, the effective potential (60) vanishes too. At $L_T = 0$ ($a=1$), the effective force (64) on the $q\bar{q}$ pairs should be zero at the threshold condition. Thus,

$$\mathscr{E}_{th} = \frac{2\pi m^2}{\sqrt{\lambda}}\sqrt{\frac{1-(\lambda^2T^4/16m^4)}{1+(16\pi^4m^4\theta^2/\lambda)}}. \tag{65}$$

Thus, we see that the effect of finite temperature is to decrease the threshold electric field. However, noncommutative effects do not mix up with the influence of finite temperature (in the sense that there are no "θT" mixed terms in the expression of the threshold electric field). Similar inference can be drawn from studying the quasinormal modes of scalar perturbations in the presence of noncommutativity as in [36] (We thank Juan F. Pedraza for bringing this to our notice.) which shows enhancement of the dissipation rate in accordance to decrement of threshold field. Also, as shown in [37, 38], the NCYM is less viscous than its commutative counterpart. It is of interest to wonder whether viscosity and threshold electric field has anything to do with each other. From (65) and (64), we have

$$\begin{aligned} \frac{dU_{T,\text{eff}}(a)}{dL_T} &= (1-R)\,\mathscr{E}_{th} - \frac{2\pi m^2}{\sqrt{\lambda}}\left(\sqrt{\frac{1-(\lambda^2T^4/16m^4)}{1+(16\pi^4m^4\theta^2/\lambda)}}\right. \\ &\quad\left.- a^2\sqrt{\frac{1-(\lambda^2T^4/16m^4a^4)}{1+(16\pi^4m^4\theta^2/\lambda)a^4}}\right). \end{aligned} \tag{66}$$

The monotonic nature of the second term w.r.t. the parameter a is quite clear in the allowed range of a. Thus, the value of $\mathscr{E}_{th}$ so found suffices to cause vacuum decay at the threshold point ($R = 1$).

4. Pair Production Rate of the Noncommutative Schwinger Effect

In this section, we would like to estimate the rate of production of $q\bar{q}$ pairs interacting with NCYM in the presence of an external electric field along a noncommutative direction. As indicated in (11), the production rate is proportional to the Wilson loop of the classical Euclidean trajectory of particles under the presence of the electric field, i.e., a circle in the $x_0 - x_3$ plane. An explicit solution of the circular string profile for the gravity dual of $\mathcal{N} = 4$ SYM is given in [39] and in [11]. Here, we state the same for later purposes.

$$x_0(t,s) = R\,\frac{\cosh s_0}{\cosh s}\cos t\,;\quad x_3(t,s) = R\frac{\cosh s_0}{\cosh s}\sin t\,;\quad u(t,s) = u_B\frac{\tanh s_0}{\tanh s}. \tag{67}$$

The solution holds true in the conformal gauge of the Polyakov action which is equivalent to the Nambu-Goto action at the classical level. In the above, R indicates the radius of the Wilson loop on the probe brane. The parameter s in one of the coordinates of the 2-dimensional string worldsheet and its value on the probe brane is given by s_0; t parametrizes the circular contour on the probe brane and thus has range $[0, 2\pi]$. Moreover, one can obtain the relation, $\sinh s_0 = 1/Ru_B$, which connects the allowed range of the worldsheet parameter to the physical quantities like mass and external electric field. For the present purpose, the relevant gravity dual is given by (27). The Polyakov action in conformal gauge looks

$$S = \frac{\sqrt{\lambda}}{4\pi}\int dtds\left[U^2\partial_a X_0\partial_a X_0 + \frac{1}{U^2}\partial_a U\partial_a U + \frac{U^2}{1+\alpha U^4}\partial_a X_3\partial_a X_3 + \cdots\right]. \tag{68}$$

Both the worldsheet and target spacetime have been continued to Euclidean signature. We have redefined $\alpha \equiv \lambda\theta^2$ and have neglected the terms involving $X_{1,2}$. The equations of motion corresponding to (68) are given by

$$2\partial_t U\partial_t X_0 + 2\partial_s U\partial_s X_0 + U\left(\partial_t^2 X_0 + \partial_s^2 X_0\right) = 0, \tag{69}$$

$$U\left(1+\alpha U^4\right)\left(\partial_t^2 X_3 + \partial_s^2 X_3\right) = 2\left(\alpha U^4 - 1\right)\left(\partial_t U\partial_t X_3 + \partial_s U\partial_s X_3\right), \tag{70}$$

$$\left(\partial_t X_0\right)^2 + \left(\partial_s X_0\right)^2 + \frac{\left(1-\alpha U^4\right)}{\left(1+\alpha U^4\right)^2}\left(\left(\partial_t X_3\right)^2 + \left(\partial_s X_3\right)^2\right) + \frac{1}{U^4}\left(\left(\partial_s U\right)^2 + \left(\partial_t U\right)^2 - U\partial_t^2 U - U\partial_s^2 U\right) = 0. \tag{71}$$

These equations are to be supplemented with the condition $X_0^2(t,s_0) + X_3^2(t,s_0) = R^2$. The set of equations in (69)–(71) form a system of coupled second-order nonlinear differential equations and in general is impossible to solve. In the context of gauge/string duality, the solution of Wilson loops in general background has been a perplexing issue. A certain way has been suggested in [40] based on employing a "circular ansatz," but it can be checked that such methods are valid only if the background has $SO(2)$ isometries in the plane of the Wilson loop (which in our case, i.e., $x_0 - x_3$ plane is absent). However, if relevant the background is a continuous parametric deformation of AdS, one can describe the string profile as $X_\mu(t,s;\sigma_i)$ where σ_i collectively indicates the deformation parameters.

Expanding in power series, $X_\mu(t,s;\sigma_i) = X_\mu(t,s;\sigma_i=0) - \sigma_i\cdot\partial_{\sigma_i}X_\mu(t,s;\sigma_i=0) + \mathcal{O}(\sigma_i^2)$, and noting that $X_\mu(t,s;\sigma_i=0)$ is the known AdS solution, the nonlinear equations become simplified. In the present context, the deformation parameter is $\sigma \equiv \alpha = \lambda\theta^2$. Crudely speaking, this amounts to treating NCYM as a perturbation over YM. Using the expansion,

$$X_0(t,s) = K(\cos\,t\;\operatorname{sech}\,s - \alpha\chi_0(t,s)),\quad X_3(t,s) = K(\sin\,t\;\operatorname{sech}\,s - \alpha\chi_3(t,s)),\quad U(t,s) = \frac{1}{K}(\coth s - \alpha\xi(t,s)). \tag{72}$$

One obtains the following equations at order $\mathcal{O}(\alpha)$:

$$\coth\,s\left(\partial_t^2\chi_0 + \partial_s^2\chi_0\right) - 2\,\sin\,t\;\operatorname{sech}\,s\,\partial_t\xi - 2\xi\cos\,t\;\operatorname{sech}^3 s - 2\;\operatorname{csch}^2 s\,\partial_s\chi_0 - 2\;\cos\,t\;\operatorname{sech}\,s\,\partial_s\xi = 0, \tag{73}$$

$$\coth s\left(\partial_t^2\chi_3 + \partial_s^2\chi_3\right) - 2\xi\sin t\operatorname{sech}^3 s + 2\cos t\operatorname{sech} s\partial_t\xi - 2\sin t\tanh s\operatorname{sech} s\partial_s\xi - 2\operatorname{csch}^2 s\partial_s\chi_3 = \frac{4}{K^4}\sin t\coth^2 s\operatorname{csch}^3 s, \tag{74}$$

$$\begin{aligned}\frac{3}{K^4}\coth^8 s\left(\operatorname{sech}^2 s\cos^2 t + \sin^2 t\tanh^2 s\operatorname{sech}^2 s\right) &= 4\coth^3 s\operatorname{sech}^2 s\left(1+\tanh^2 s\right)\xi - \coth s\\ &\times\left(\partial_t^2\xi + \partial_s^2\xi + 2\frac{\operatorname{csch}^2}{\coth s}\partial_s\xi + 2\operatorname{csch}^2 s\xi\right)\\ &- \coth^4 s\operatorname{sech} s(\sin t\partial_t\chi_0 + \cos t\tanh s\partial_s\chi_0\\ &+ \sin t\tanh s\partial_s\chi_3 - \cos t\partial_s\chi_3).\end{aligned} \tag{75}$$

In deriving the above, (69)–(71) has been linearized using

(This is possible because in the present case, we have an upper bound of $U = u_B$; in using the binomial expansion, we have assumed $\lambda\theta^2 u_B^4$to be small.) $1/(1+\alpha U^4) \approx 1 - \alpha U^4$, and then, (72) has been used keeping in mind that terms of $\mathcal{O}(\alpha^0)$ are AdS equations which are automatically zero. Moreover, we have assumed $\alpha U^4(t,s) \approx \alpha/K^4 \coth^4 s$ up to first order in α. Equations (73)–(75) though being simplified than before are still daunting. Using the ansatz $\xi(t,s) = \xi(s)$, $\chi_0(t,s) = \chi_0(s)\cos t$, $\chi_3(t,s) = \chi_3(s)\sin t$ in (74) and (75), one has

$$\partial_s\chi_0 = \frac{1}{2}\sinh^2 s\coth s\left(\partial_s^2\chi_0 - \chi_0\right) - \xi\sinh^2 s\,\mathrm{sech}^3 s - \mathrm{sech}\,s\tanh s\sinh^2 s\,\partial_s\xi, \tag{76}$$

$$\partial_s\chi_3 = \frac{1}{2}\sinh^2 s\coth s\left(\partial_s^2\chi_3 - \chi_3\right) - \xi\sinh^2 s\,\mathrm{sech}^3 s - \mathrm{sech}\,s\tanh s\sinh^2 s\,\partial_s\xi - \frac{2}{K^4}\coth^2 s csch\,s. \tag{77}$$

Since the set (73)–(75) are coupled differential equations, the solution of the first two has to satisfy the other one. Substituting (76) and (77) in (75) and noting that the resulting equation has to be satisfied for all values of parameter t, one obtains the following three equations:

$$\tanh s\sinh s\,\partial_s^2\chi_0 - 2\,\mathrm{sech}\,s\,\chi_3 - \tanh s\sinh s\,\chi_0 + \frac{3}{K^4}\coth^2 s\,\mathrm{csch}^2 s = 0, \tag{78}$$

$$\tanh s\sinh s\,\partial_s^2\chi_3 - 2\,\mathrm{sech}\,s\,\chi_0 - \tanh s\sinh s\,\chi_3 - \frac{1}{K^4}^{2}\mathrm{csch}\,s = 0, \tag{79}$$

$$\coth s\,\partial_s^2\xi - 2\xi\coth s\,\mathrm{csch}^2 s\left(1 + 3\tanh^2 s\right) + 2\left(\mathrm{csch}^2 s - 1\right)\partial_s\xi = 0. \tag{80}$$

It can be checked that (80) has no real solution, furthermore, (80) being a linear equation permits a solution of the form $\xi = 0$. Thus, we are left with first two equations (78) and (79) which are coupled differential equations themselves. To simplify those, we define the following variables whose significance is rather obscure.

$$\chi_+(s) = \chi_0(s) + \chi_3(s)\,; \qquad \chi_-(s) = \chi_0(s) - \chi_3(s). \tag{81}$$

In terms of the above, one has

$$\partial_s^2\chi_+ - \chi_+ - \mathrm{csch}^2 s\chi_+ + \frac{\mathrm{csch}^3 s}{K^4}\coth s\left(3\coth^2 s - 1\right), \tag{82}$$

$$\partial_s^2\chi_- - \chi_- + \mathrm{csch}^2 s\chi_- + \frac{\mathrm{csch}^3 s}{K^4}\coth s\left(3\coth^2 s - 1\right). \tag{83}$$

Digressing a bit from the main discussion, let us see the first-order correction to the on-shell action in light of the perturbation theory set up. From the decomposition (72), one has up to $\mathcal{O}(\alpha)$

$$\begin{aligned}
(\partial_t X_0)^2 &= K^2\sin^2 t\left(\mathrm{sech}^2 s - 2\alpha\,\mathrm{sech}\,s\,\chi_0\right),\\
(\partial_t X_3)^2 &= K^2\cos^2 t\left(\mathrm{sech}^2 s - 2\alpha\,\mathrm{sech}\,s\,\chi_3\right),\\
(\partial_s X_0)^2 &= K^2\cos^2 t\left(\mathrm{sech}^2 s\tanh^2 s + 2\alpha\,\mathrm{sech}\,s\tanh s\,\partial_s\chi_0\right),\\
(\partial_s X_3)^2 &= K^2\sin^2 t\left(\mathrm{sech}^2 s\tanh^2 s + 2\alpha\,\mathrm{sech}\,s\tanh s\,\partial_s\chi_3\right).
\end{aligned} \tag{84}$$

Using the above in the Polyakov action in the presence of the NC dual (68) and approximating $1/(1 + (\alpha/K^4)\alpha/K^4 \coth^4 s) \approx 1 - (\alpha/K^4)\coth^4 s$, one has in the first order of the effective parameter

$$\mathcal{L}_{\text{on-shell}} = 2\,\mathrm{csch}^2 s - 2\alpha\frac{\cosh s}{\sinh^2 s}\left[\sin^2 t\left\{\chi_0 - \tanh s\,\partial_s\chi_3 - \frac{\coth^2 s}{2K^4}\mathrm{sech}\,s\right\} + \cos^2 t\left\{\chi_3 - \tanh s\,\partial_s\chi_0 - \frac{\coth^4 s}{2K^4}\mathrm{sech}\,s\right\}\right]. \tag{85}$$

Using equations (78) and (79) and the fact that $\xi(s) = 0$, one can reduce the first-order correction to above expression to

$$\begin{aligned}
\delta_\alpha\mathcal{L}_{\text{on-shell}} &= -\alpha\cosh s\left[\sin^2 t\left\{2\,\mathrm{csch}^2 s\chi_0 + \frac{3}{K^4}\coth^2 s\,\mathrm{sech}\,s\,\mathrm{csch}^2 s - \partial_s^2\chi_3 + \chi_3\right\} + \cos^2 t\left\{2\,\mathrm{csch}^2 s\,\chi_3 - \frac{1}{K^4}\coth^4 s\,\mathrm{sech}\,s\,\mathrm{csch}^2 s - \partial_s^2\chi_0 + \chi_0\right\}\right]\\
&= -\frac{2\alpha}{K^4}\coth^2\,\mathrm{csch}^2 s\left[1 + \mathrm{csch}^2 s\cos^2 t\right].
\end{aligned} \tag{86}$$

In the last line, (78) and (79) have been put to use. Thus, we see that the equations of motions alone determine the first-order correction of the on-shell action from the commutative counterpart. In the context of holographic entanglement entropy, similar methods have been presented in [40]. For finding out the limit of the integration and its connection to physical variables, one has to solve the equations of motion. Returning to our main discussion, the real part of the solution of (82) is

$$\chi_+(s) = \chi_0(s) + \chi_3(s) = -\frac{1}{6K^4}\coth^4 s\,\mathrm{sech}\,s. \tag{87}$$

Equation (83) which dictates the deviation of the circular symmetry cannot be solved by analytical means. It is worthwhile to note that (83) is not a linear equation and does not admit a solution $\chi_-(s) = 0$. However, since it is a second-order differential equation with two constants, one can redefine them to set $\chi_-(s = s_0) = 0$ for a specific value s_0, which is essentially imposing the boundary condition of the loop being circular at the probe brane. There exist no known methods to solve a generic second-order partial differential equation. We do not claim that the first derivative of χ_- is zero at s_0. Thus, at the point given by $s = s_0$, the profile is circular and the variable χ_+ is twice the radius of the loop(R). Putting the above in mathematical language, from (72) and (87)

$$R = K\left(\text{sech } s_0 + \frac{\alpha}{12K^4}\coth^4 s_0 \text{ sech } s_0\right), \qquad u_B = \frac{1}{K}\coth s_0. \tag{88}$$

From the above, one gets

$$Ru_B = \text{csch } s_0\left(1 + \frac{\alpha}{12}u_B^4\right). \tag{89}$$

This relation serves to define the integration limits and also connects the on-shell value of the action to physical parameters. From the above, one has in the first order of the noncommutative parameter

$$\coth s_0 = \sqrt{1 + R^2u_B^2\left(1 - \frac{\alpha}{6}u_B^4\right)}. \tag{90}$$

In presence of an external electric field in the x_3 direction, the on-shell value of the action gets modified to ($U(1)$ gauge fields contribute to the string Lagrangian via a boundary term. For constant electromagnetic field, the string equations of motion are unchanged but the boundary conditions are altered (Robin). For inhomogeneous fields, this is not the case. The Schwinger effect for inhomogeneous fields has been explored via holographic methods in [41].)

$$S_{\text{on-shell}} = \sqrt{\lambda}\left[\{(1-3\eta)\coth s_0 - 1\} - 5\eta\frac{1}{\coth s_0} + 8\eta\frac{1}{\coth^4 s_0} - \mathscr{E}\,\text{csch}^2 s_0\right]. \tag{91}$$

In the above, we replaced the constants by hyperbolic functions (88) and have defined $\eta = (\alpha/30)u_B^4 \equiv (\lambda\theta^2/30)u_B^4$ and $\mathscr{E} = (\pi/\sqrt{\lambda}u_B^2)(1 + (5/2)\eta)^2 E$. Note that the dependence on the radius (R) is now encoded in the hyperbolic functions themselves. Quite similar to arguments in [11, 42] at a large value of R (large E), the production rate (11), (30), and (91) is dominated by $\Gamma \sim \exp(\sqrt{\lambda}R^2E)$ similar to the phase (in potential analysis) when V_{sch} does not permit a tunneling barrier. However, for small R (πR^2E not dominating the other terms), the approximate production rate is

$$\Gamma \sim \exp(-S_{\text{on-shell}}) \sim \exp\left(-\frac{\sqrt{\lambda}u_B^2}{2}(1-5\eta)(1-30\eta)R^2 + \pi ER^2 + \mathscr{O}(R^4)\right). \tag{92}$$

It is evident that as R varies, one moves from a tunneling or damped production phase (when the first term in (92) dominates) to a spontaneous production phase (when the second term dominates). This is quite synonymous to the potential analysis with the identification $\Gamma \sim \exp(-V_{\text{effective}})$. The potential barrier vanishes when both terms in (92) cancel each other which happens at

$$E_{\text{threshold}} \sim \frac{\sqrt{\lambda}}{2\pi}u_B^2(1-5\eta)(1-30\eta) \sim \frac{2\pi m^2}{\sqrt{\lambda}}\left(1 - \frac{56}{3}\frac{\pi^4m^4\theta^2}{\lambda}\right). \tag{93}$$

To get the production rate, one needs to extremize the on-shell action with respect to R for reasons mentioned before. Instead of extremizing w.r.t. R, one can extremize w.r.t. s_0; however, doing so, one is left to solve a sextic equation. To simplify the situation, note that the value of $\coth s_0$ is proportional to the mass of the quark (W boson) via the relation derived earlier. Thus, for heavy mass, (actually $\lambda\theta^2m^4/m^6 \ll 1$), the contribution of the second and third term of (91) is negligible. Under those circumstances, one has

$$S_{\text{on-shell}} \approx \sqrt{\lambda}\left[(1-3\eta)\coth s_0 - 1 - \mathscr{E}^2 s_0\right]. \tag{94}$$

Extremizing the above w.r.t. s_0 (i.e., R), one is lead to $\coth s_0 = (1-3\eta)/2\mathscr{E}$ which is a condition connecting R and E. From the relation thus obtained, the on-shell value of the action is

$$S_{\text{on-shell}} = \frac{\sqrt{\lambda}(1-3\eta)}{2}\left[\frac{(1-3\eta)(1-5\eta)\sqrt{\lambda}u_B^2}{2\pi E} - \frac{2}{(1-3\eta)} + \frac{2\pi E}{(1-3\eta)(1-5\eta)\sqrt{\lambda}u_B^2}\right]. \tag{95}$$

Thus, the production rate which is proportional to the (negative) exponential of the on-shell action (11), (30), and (95) is given by

$$\Gamma \sim \exp\left[-\frac{\sqrt{\lambda}}{2}\left(1 - \frac{8\pi^4m^4\theta^2}{5\lambda}\right)\left\{\sqrt{\frac{\mathscr{E}_T}{E}} - \sqrt{\frac{E}{\mathscr{E}_T}}\right\}^2 + \frac{8\pi^4m^4\theta^2}{5\sqrt{\lambda}}\right], \tag{96}$$

where we have restored the physical parameters via the relation $\eta = 8\pi^4m^4\theta^2/15\lambda$. In the above, the threshold electric field is given by

$$\mathscr{E}_T = \frac{2\pi m^2}{\sqrt{\lambda}}\left(1 - \frac{8}{5}\frac{\pi^4m^4\theta^2}{\lambda}\right)\left(1 - \frac{8}{3}\frac{\pi^4m^4\theta^2}{\lambda}\right) \approx \frac{2\pi m^2}{\sqrt{\lambda}}\left(1 - \frac{64}{15}\frac{\pi^4m^4\theta^2}{\lambda}\right). \tag{97}$$

At low electric field, $E \ll \mathscr{E}_T$, the second term in (96) ceases to contribute and one is left with

$$\Gamma \sim \exp\left[-\frac{\pi m^2}{E}\left(1 - \frac{88\pi^4m^4\theta^2}{15\lambda}\right) + \left(\frac{8\pi^4m^4\theta^2}{5\sqrt{\lambda}}\right)\right]. \tag{98}$$

One can compare (98) to the result of [9], and both of them show the same pattern with the identification of $\overleftarrow{B} \sim \theta$ where $\overleftarrow{B}$ indicates an external magnetic field in [9]. In the presence of strong magnetic fields, commutativity is lost and the theory is described by noncommutative physics [43]. We come up with "three" values of threshold electric field in (46), (93), and (97). The reason they do not match is because the latter two values have been found via perturbative methods while the former (46) by exact methods. Moreover, one should notice that (93) is estimated from (92) which itself is valid for small R, unlike (97), which is derived at the limit of large quark mass (where the assumption on R is relaxed). However, all of them point towards the same fact, *threshold electric field is decreased due to noncommutativity*. A reason for concern may be the extra (θ dependent) term in (96) and (98) which is independent of the electric field. We believe this is an effect of our simplification of solving a quadratic equation instead of a sextic one (see above).

A reason for concern: one can also find out the "threshold" electric field from the DBI action (by claiming that the DBI action should be real for all allowed values of the electric field) [44, 45]. However, such analysis shows that the threshold electric field is the same as that of the commutative case irrespective of whether the applied electric field is parallel or perpendicular to the noncommutative directions contrary to our findings. This issue is not clear to us at the moment.

5. Conclusions

In this paper, we have performed an interquark potential analysis to find the effective potential barrier in the presence of an external electric field in noncommutative gauge theory. From the same, we have shown that the threshold electric field is decreased from its commutative counterpart. In the presence of noncommutativity, there exist strong repulsive forces between the particles at short distances, i.e., the Coulombic interaction develops a short distance repulsive correction. This implies the electrostatic potential energy needed to tear out the virtual particles is less than usual explaining the result found. We have also argued that noncommutativity does not lead to confinement as at large distances, the behavior of the potential is essentially Coulombic as demonstrated. We also have found out the thermal corrections to the above and have seen that finite temperature effects do not entangle with (space-space) noncommutative ones as expected which also can be seen by putting the values of (27) and (28) in [46]. Moreover, we have also perturbatively computed the corrections to the circular Wilson loop over the known commutative result in the first order of the noncommutative deformation parameter, and hence, the decay rate has been found out from which the decrement of the threshold value is also clear. To the best of our knowledge, such an analysis for the noncommutative holographic Schwinger effect is new. At low electric field, our result shares the same pattern with that of [9] for noncommutative $U(1)$ gauge theory.

Data Availability

No data is used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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References

[1] J. Schwinger, "On gauge invariance and vacuum polarization," *Physics Review*, vol. 82, no. 5, pp. 664–679, 1951.

[2] F. Gelis and N. Tanji, "Schwinger mechanism revisited," *Progress in Particle and Nuclear Physics*, vol. 87, pp. 1–49, 2016.

[3] H. S. Snyder, "Quantized space-time," *Physics Review*, vol. 71, no. 1, pp. 38–41, 1947.

[4] R. Szabo, "Quantum field theory on noncommutative spaces," *Physics Reports*, vol. 378, no. 4, pp. 207–299, 2003.

[5] V. Schomerus, "D-branes and deformation quantization," *Journal of High Energy Physics*, vol. 1999, no. 6, 1999.

[6] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *Journal of High Energy Physics*, vol. 1999, no. 9, 1999.

[7] J. Madore, *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*, Cambridge University Press, 1999.

[8] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane, and T. Okamoto, "Noncommutative field theory and Lorentz violation," *Physical Review Letters*, vol. 87, no. 14, article 141601, 2001.

[9] N. Chair and M. M. Sheikh-Jabbari, "Pair production by a constant external field in noncommutative QED," *Physics Letters B*, vol. 504, no. 1-2, pp. 141–146, 2001.

[10] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, "Large *N* field theories, string theory and gravity," *Physics Reports*, vol. 323, no. 3-4, pp. 183–386, 2000.

[11] G. W. Semeno and K. Zarembo, "Holographic Schwinger effect," *Physical Review Letters*, vol. 107, 2011.

[12] Y. Sato and K. Yoshida, "Holographic Schwinger effect in confining phase," *Journal of High Energy Physics*, vol. 2013, no. 9, 2013.

[13] J. Maldacena, "The large N limit of superconformal field theories and supergravity," *International Journal of Theoretical Physics*, vol. 38, no. 4, pp. 1113–1133, 1999.

[14] N. Drukker, D. J. Gross, and H. Ooguri, "Wilson loops and minimal surfaces," *Physical Review D*, vol. 60, no. 12, 1999.

[15] I. K. Affleck, O. Alvarez, and N. S. Manton, "Pair production at strong coupling in weak external fields," *Nuclear Physics B*, vol. 197, no. 3, pp. 509–519, 1982.

[16] Y. Sato and K. Yoshida, "Holographic description of the Schwinger effect in electric and magnetic field," *Journal of High Energy Physics*, vol. 2013, no. 4, 2013.

[17] J. Gordon and G. W. Semenoff, "World-line instantons and the Schwinger effect as a Wentzel–Kramers–Brillouin exact path integral," *Journal of Mathematical Physics*, vol. 56, no. 2, p. 022111, 2015.

[18] J. Gordon and G. W. Semenoff, "Worldsheet Instantons and the amplitude for string pair production in an external field

as a WKB exact functional integral," *Journal of High Energy Physics*, vol. 2018, no. 5, 2018.

[19] C. Itzykson and J. Zuber, *Quantum Field Theory*, International Series In Pure and Applied Physics, McGraw-Hill, New York, 1980.

[20] C. Schubert, "Perturbative quantum field theory in the string-inspired formalism," *Physics Reports*, vol. 355, no. 2-3, pp. 73–234, 2001.

[21] C.-S. Chu and P.-M. Ho, "Non-commutative open string and D-brane," *Nuclear Physics B*, vol. 550, no. 1-2, pp. 151–168, 1999.

[22] R. Blumenhagen, D. Lust, and S. Theisen, *Basic Concepts of String Theory*, Theoretical and Mathematical Physics, Springer, Heidelberg, Germany, 2013.

[23] M. M. Sheikh-Jabbari, "More on mixed boundary conditions and D-branes bound states," *Physics Letters B*, vol. 425, no. 1-2, pp. 48–54, 1998.

[24] F. Ardalan, H. Arfaei, and M. M. Sheikh-Jabbari, "Noncommutative geometry from strings and branes," *Journal of High Energy Physics*, vol. 1999, no. 2, 1999.

[25] A. Bilal, C.-S. Chu, and R. Russo, "String theory and noncommutative field theories at one loop," *Nuclear Physics B*, vol. 582, no. 1-3, pp. 65–94, 2000.

[26] J. M. Maldacena and J. G. Russo, "LargeNlimit of non-commutative gauge theories," *Journal of High Energy Physics*, vol. 1999, no. 9, 1999.

[27] M. Alishahiha, Y. Oz, and M. M. Sheikh-Jabbari, "Supergravity and largeNnoncommutative field theories," *Journal of High Energy Physics*, vol. 1999, no. 11, 1999.

[28] A. Hashimoto and N. Itzhaki, "Non-commutative Yang-Mills and the AdS/CFT correspondence," *Physics Letters B*, vol. 465, no. 1-4, pp. 142–147, 1999.

[29] Y. Sato and K. Yoshida, "Potential analysis in holographic Schwinger effect," *Journal of High Energy Physics*, vol. 2013, no. 8, 2013.

[30] W. Fischler, P. H. Nguyen, J. F. Pedraza, and W. Tangarife, "Holographic Schwinger effect in de Sitter space," *Physical Review D*, vol. 91, no. 8, 2015.

[31] J. Maldacena, "Wilson loops in LargeNField theories," *Physical Review Letters*, vol. 80, no. 22, pp. 4859–4862, 1998.

[32] S.-J. Rey and J.-T. Yee, "Macroscopic strings as heavy quarks: large-N gauge theory and anti-de Sitter supergravity," *The European Physical Journal C*, vol. 22, no. 2, pp. 379–394, 2001.

[33] A. Dhar and Y. Kitazawa, "Wilson loops in strongly coupled noncommutative gauge theories," *Physical Review D*, vol. 63, no. 12, 2001.

[34] Y. Kinar, E. Schreiber, and J. Sonnenschein, "$Q\bar{Q}$ potential from strings in curved space-time - classical results," *Nuclear Physics B*, vol. 566, no. 1-2, pp. 103–125, 2000.

[35] M. Ghodrati, "Schwinger effect and entanglement entropy in confining geometries," *Physical Review D*, vol. 92, no. 6, 2015.

[36] M. Edalati, W. Fischler, J. F. Pedraza, and W. T. Garcia, "Fast scramblers and non-commutative gauge theories," *Journal of High Energy Physics*, vol. 2012, no. 7, 2012.

[37] W. Fischler, J. F. Pedraza, and W. Tangarife Garcia, "Holographic Brownian motion in magnetic environments," *Journal of High Energy Physics*, vol. 2012, no. 12, 2012.

[38] T. Matsuo, D. Tomino, and W.-Y. Wen, "Drag force in SYM plasma with B field from AdS/CFT," *Journal of High Energy Physics*, vol. 2006, no. 10, 2006.

[39] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, "Operator product expansion for Wilson loops and surfaces in the largeNlimit," *Physics Review D*, vol. 59, no. 10, 1999.

[40] D. Allahbakhshi, M. Alishahiha, and A. Naseh, "Entanglement thermodynamics," *Journal of High Energy Physics*, vol. 2013, 2013.

[41] C. Lan, Y.-F. Wang, H. Geng, and A. Andreev, "Holographic Schwinger effect of dynamic fields," http://arxiv.org/abs/1811.11712.

[42] S. Bolognesi, F. Kiefer, and E. Rabinovici, "Comments on Critical electric and magnetic fields from holography," *Journal of High Energy Physics*, vol. 2013, no. 1, 2013.

[43] V. A. Miransky and I. A. Shovkovy, "Quantum field theory in a magnetic field: from quantum chromodynamics to graphene and Dirac semimetals," *Physics Reports*, vol. 576, pp. 1–209, 2015.

[44] X. Wu, "Notes on holographic Schwinger effect," *Journal of High Energy Physics*, vol. 2015, no. 9, 2015.

[45] K. Hashimoto and T. Oka, "Vacuum instability in electric fields via AdS/CFT: Euler-Heisenberg Lagrangian and Planckian thermalization," *Journal of High Energy Physics*, vol. 2013, no. 10, 2013.

[46] Y. Sato and K. Yoshida, "Universal aspects of holographic Schwinger effect in general backgrounds," *Journal of High Energy Physics*, vol. 2013, no. 12, 2013.