

## Research Article

# Hořava-Lifshitz Scalar Field Cosmology: Classical and Quantum Viewpoints

Fatimah Tavakoli,<sup>1</sup> Babak Vakili ,<sup>1</sup> and Hossein Ardehali<sup>2</sup>

<sup>1</sup>Department of Physics, Central Tehran Branch, Islamic Azad University, Tehran, Iran

<sup>2</sup>Department of Physics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Correspondence should be addressed to Babak Vakili; [b.vakili@iauctb.ac.ir](mailto:b.vakili@iauctb.ac.ir)

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In this paper, we study a projectable Hořava-Lifshitz cosmology without the detailed balance condition minimally coupled to a nonlinear self-coupling scalar field. In the minisuperspace framework, the super-Hamiltonian of the presented model is constructed by means of which some classical solutions for the scale factor and scalar field are obtained. Since these solutions exhibit various types of singularities, we came up with the quantization of the model in the context of the Wheeler-DeWitt approach of quantum cosmology. The resulting quantum wave functions are then used to investigate the possibility of the avoidance of classical singularities due to quantum effects which show themselves important near these singularities.

## 1. Introduction

In recent years, a new gravitation theory based on anisotropic scaling of the space  $x$  and time  $t$  was presented by Hořava. Since the methods used in this gravitation theory are similar to the Lifshitz work on the second-order phase transition in solid-state physics, it is commonly called Hořava-Lifshitz (HL) theory of gravity [1–4]. In general, HL gravity is a generalization of general relativity (GR) at a high-energy ultraviolet (UV) regime and reduces to standard GR in the low-energy infrared (IR) limit. However, unlike another candidates of quantum gravity, the issue of the Lorentz symmetry breaking at high energies is described somehow in a different way. Here, the well-known phenomenon of Lorentz symmetry breaking will be expressed by a Lifshitz-like process in solid-state physics. This is based on the anisotropic scaling between space and time as

$$t \longrightarrow b^z t, x \longrightarrow bx, \quad (1)$$

where  $b$  is a scaling parameter and  $z$  is a dynamical critical exponent. It is clear that  $z = 1$  corresponds to the standard relativistic scale invariance with the Lorentz symmetry.

Indeed, with  $z = 1$ , the theory falls within its IR limit. However, different values of  $z$  correspond to different theories, for instance, what is proposed in [1, 2] as the UV gravitational theory requires  $z = 3$ . In order to better represent the asymmetry of space and time in HL theory, we write the space-time metric in its ADM form, that is,

$$g_{\mu\nu}(t, x) = \begin{pmatrix} -N^2(t, x) + N_a(t, x)N^a(t, x) & N_b(t, x) \\ N_a(t, x) & h_{ab}(t, x) \end{pmatrix}, \quad (2)$$

where  $N(t, x)$  is the lapse function,  $N^a(t, x)$  are the components of the shift vector, and  $h_{ab}(t, x)$  is the spacial metric. There are two classes of HL theories depending on whether the lapse function is a function only of  $t$ , for which the theory is called projectable, or of  $(t, x)$ , for which we have a nonprojectable theory. Since in cosmological settings, the lapse function usually is chosen only as a function of time, the corresponding HL cosmologies are projectable [5–7]. In more general cases however, one may consider the lapse function as a function of both  $t$  and  $x$  to get a nonprojectable theory (see [8, 9]). At first glance, it may seem that imposing

the lapse function to be just a function of time is a serious restriction. However, it should be noted that in the framework of this assumption, the classical Hamiltonian constraint is no longer a local constraint, but its integral over all spatial coordinates should be done, which means that we do not have a local energy conservation. In [10], it is shown that this procedure yields classical solutions to the IR limit of HL gravity which are equivalent to Friedmann equations with an additional term of the cold dark matter type. On the other hand, in homogeneous models like the Robertson-Walker metric, such spatial integrals are simply the spatial volume of the space and thus the above-mentioned dark dust constant must vanish [11, 12]. In summary, it is worth noting that although almost all physically important solutions of the Einstein equations like Schwarzschild, Reissner-Nordström, Kerr, and Friedmann-Lemaitre-Robertson-Walker space times can be cast into the projectable form by suitable choice of coordinates, most of the results, in principle, may be extended to the nonprojectable case through a challenging but straightforward calculation [13, 14].

Another thing to note about the HL theory is the form of its action which in general consists of two kinetic and potential terms. Its kinetic term  $S_K$  is nothing other than what comes from the Einstein-Hilbert action. The form for the potential term is  $S_V = \int d^4x \sqrt{-g} V[h_{ab}]$ , where  $V$  is a scalar function that depends only on the spacial metric  $h_{ab}$  and its spacial derivatives. Among the very different possible combinations that can be constructed using the three-metric scalars [5], Hořava considered a special form in  $z = 3$  theory known as “detailed balance condition,” in which the potential is a combination of the terms  $\nabla_a R_{bc} \nabla^a R^{bc}$ ,  $\nabla_a R_{bc} \nabla^b R^{ac}$ ,  $\nabla_a R \nabla^a R$  [2, 3]. Here, we do not go into the details of this issue. The detailed balanced theories show simpler quantum behavior because they have simpler renormalization properties. However, as shown in [11, 12], if one relaxes this condition, the resulting action with extra allowed terms is well-behaved enough to recover the models with detailed balance. Another feature of HL theory is its known inconsistency problems such as its instabilities, ghost scalar modes, and the strong coupling problem. Indeed, by perturbation of this theory around its IR regime, one can show that it suffers from some instabilities and fine-tunings that may not be removed by usual tricks such as analytic continuation. Since our study in this article is done at the background level, such issues are beyond our discussion. However, a detailed review of this topic can be found in [15]. On the other hand, there are some extensions of the initial version of the HL gravity theory that deal with such problems. Some of these are as follows: [16], in which a projectable  $U(1)$  symmetric soft-breaking detailed balance condition model is considered and it is shown that the resulting theory displays anisotropic scaling at short distances while almost all features of GR are recovered at long distances. The nonprojectable model without detailed balance condition is studied in [9], where it is proven that only nonprojectable model is free from instabilities and strong coupling. The  $U(1)$  symmetric nonprojectable version of the HL gravity is studied in [17, 18], in which a coupling of the theory with a scalar field is also considered and it is shown that all the problems that the original theory suffers

from will disappear. Finally, a progress report around all of the above-mentioned issues has been reviewed in [19].

In this paper, we consider a Friedmann-Robertson-Walker (FRW) cosmological model coupled to a self-interacting scalar field, in the framework of a projectable HL gravity without detailed balance condition. The basis of our work to deal with this issue is through its representation with minisuperspace variables. Minisuperspace formulation of classical and quantum HL cosmology is studied in some works (see, for instance, [5, 6] and [20–23]). Also, quantization of the HL theory without restriction to a cosmological background is investigated, for instance, in [24], in which the quantization of two-dimensional HL theory without the projectability condition is considered, and [25], where a  $(1 + 1)$ -dimensional projectable HL gravity is quantized.

Here, we first construct a suitable form for the HL action and then will add a self-coupling scalar field to it. For the flat FRW model and in some special cases, the classical solutions are presented and their singularities are investigated. We then construct the corresponding quantum cosmology based on the canonical approach of the Wheeler-DeWitt (WDW) theory to see how things may change their behavior if the quantum mechanical considerations come into the model.

## 2. The Model Outline

To study the FRW cosmology within the framework of HL gravity, let us start by its geometric structure in which in a quasispherical polar coordinate, the space time metric is assumed to be

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi) \right], \quad (3)$$

where  $N(t)$  is the lapse function,  $a(t)$  is the scale factor, and  $k = 1, 0,$  and  $-1$  correspond to the closed, flat, and open universe, respectively. In terms of the ADM variables, the above metric takes the form

$$g_{\mu\nu}(t, x) = \begin{pmatrix} -N^2(t) & 0 \\ 0 & h_{ab} \end{pmatrix}, \quad (4)$$

where

$$h_{ab} = a^2(t) \text{diag} \left( \frac{1}{1 - kr^2}, r^2, r^2 \sin^2\vartheta \right) \quad (5)$$

is the intrinsic metric induced on the spatial 3-dimensional hypersurfaces. The gravitational part of the HL action, without the detailed balance condition, is given by  $S_{\text{HL}} = S_K + S_V$ , where  $S_K$  is its kinetic part:

$$S_K \sim \int dt d^3 \mathbf{x} N \sqrt{h} (K_{ab} K^{ab} - \lambda K^2), \quad (6)$$

where  $h$  is the determinant of  $h_{ab}$  and  $\lambda$  is a correction constant to the usual GR due to HL theory. Also,  $K_{ab}$  is the extrinsic curvature (with trace  $K$ ) defined as

$$K_{ab} = \frac{1}{2N} \left( N_{ab} + N_{b|a} - \frac{\partial h_{ab}}{\partial t} \right), \quad (7)$$

where  $N_{ab}$  denotes the covariant derivative with respect to  $h_{ab}$ . Since for the FRW metric, all components of the shift vector are zero, a simple calculation based on the above definition results in  $K_{ab}K^{ab} = 3\dot{a}^2/N^2a^2$  and  $K = -3\dot{a}/Na$ , where a dot represents differentiation with respect to  $t$ . Going back to the action, its potential part is in the form

$$S_V = - \int dt d^3 \mathbf{x} N \sqrt{h} V[h_{ij}]. \quad (8)$$

According to the relation (1) and because of the anisotropic scaling of space and time coordinates, their dimensions are different as  $[\mathbf{x}] = [\kappa]^{-1}$  and  $[t] = [\kappa]^{-z}$ , where the  $[\kappa]$  is a symbol of dimension of momentum. In this sense, the dimension of the metric, lapse function, and shift vector will be  $[\gamma_{ij}] = [N] = 1$  and  $[N^i] = [\kappa]^{z-1}$ . Therefore, the potential term in a three-dimensional space has the dimension  $[V[h_{ij}]] = [\kappa]^{z+3}$ . So, according to such a dimensional analysis, one may argue that for special case  $z = 3$ , the potential  $V[h_{ij}]$  consists of the following terms of the Ricci tensor, Ricci scalar, and their covariant derivatives of dimension  $[\kappa]^6$  [26]

$$\begin{aligned} V[h_{ij}] = & g_0 \zeta^6 + g_1 \zeta^4 R + g_2 \zeta^2 R^2 + g_3 \zeta^2 R_{ij} R^{ij} \\ & + g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R_{ij} R^{jk} R_k^i \\ & + g_7 R \nabla^2 R + g_8 \nabla_i R_{jk} \nabla^i R^{jk}, \end{aligned} \quad (9)$$

where  $g_i$  ( $i = 0, \dots, 8$ ) are dimensionless coupling constants coming from HL correction to usual GR and  $\zeta$  with dimension  $[\zeta] = [\kappa]$  is introduced to make the constants  $g_i$  dimensionless. Under these conditions, the gravitational part of the HL theory that we shall consider hereafter has the following form [11, 12, 21, 26–28]:

$$\begin{aligned} S_{\text{HL}} = & \frac{M_{\text{PL}}^2}{2} \int_{\mathcal{M}} dt d^3 \mathbf{x} N \sqrt{h} \left[ K_{ij} K^{ij} - \lambda K^2 + R - 2\Lambda \right. \\ & - \frac{g_2}{M_{\text{PL}}^2} R^2 - \frac{g_3}{M_{\text{PL}}^2} R_{ij} R^{ij} - \frac{g_4}{M_{\text{PL}}^4} R^3 - \frac{g_5}{M_{\text{PL}}^4} R R_{ij} R^{ij} \\ & \left. - \frac{g_6}{M_{\text{PL}}^4} R_{ij} R^{jk} R_k^i - \frac{g_7}{M_{\text{PL}}^4} R \nabla^2 R - \frac{g_8}{M_{\text{PL}}^4} \nabla_i R_{jk} \nabla^i R^{jk} \right], \end{aligned} \quad (10)$$

in which  $M_{\text{PL}} = 1/\sqrt{8\pi G}$  and we have set  $c = 1$ ,  $\zeta = 1$ ,  $\Lambda = g_0 M_{\text{PL}}^2/2$ , and  $g_1 = -1$ .

Now, let us consider a scalar field minimally coupled to gravity but has a nonlinear interaction with itself by a coupling function  $F(\phi)$  [29]. The action of such a scalar field may be written as

$$S_\phi = \int_{\mathcal{M}} d^4 \mathbf{x} \sqrt{-g} F(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (11)$$

The existence of a scalar field in a gravitational theory can address many issues in cosmology such as spatially flat and accelerated expanding universe at the present time, inflation, dark matter, and dark energy. The action of the scalar field considered here has the same form as that in usual cosmological models with general covariance. However, in the HL gravity, the Lorentz symmetry is broken in such a way that various higher spatial derivatives will appear in the gravitational part of the action. Therefore, one expects this feature to be considered when constructing the action of the scalar field. This means that we may be able to add higher spatial derivatives of the scalar field into its action. One of the possible types of such actions for the scalar field is presented in [30] as

$$S_\phi = \int d^4 \mathbf{x} \sqrt{-g} N \left[ \frac{1}{N^2} \left( \dot{\phi} - N^i \partial_i \phi \right)^2 - V(\partial_i \phi, \phi) \right], \quad (12)$$

where the potential function  $V$  can in general contain arbitrary combinations of  $\phi$  and its spatial derivatives. However, as emphasized in [30], in homogeneous and isotropic cosmological settings like the FRW metric, we have  $N^i = 0$  and the cosmological ansatz for the scalar field is  $\phi = \phi(t)$ . In such cases since  $\partial_i \phi = 0$ , the function  $V$  in the scalar action reduces effectively to a usual potential that vanishes for a free scalar field. In this respect, the action (11) we presented for a self-interacting scalar field in the theory appears to be based on physical grounds.

The total action may now be written by adding the HL and scalar field actions as  $S = S_{\text{HL}} + S_\phi = \int dt L[a, \phi, \dot{a}, \dot{\phi}]$ . Having at hand the actions (10) and (11), by substituting the metric (3) into them, we are led to the following effective Lagrangian in terms of the minisuperspace variables  $(a(t), \phi(t))$  (the Planck mass can be absorbed in a time term and so we may rescale the time as  $t \rightarrow t/t_{\text{PL}}$ , where  $t_{\text{PL}} = 1/M_{\text{PL}}$  is the Planck time. In this sense, in what follows, by  $t$ , we mean the dimensionless quantity  $t/t_{\text{PL}}$ . In particular, the figures are plotted in terms of this quantity):

$$\begin{aligned} L[a, \phi, \dot{a}, \dot{\phi}] = & N \left( -\frac{a\dot{a}^2}{N^2} + g_c a - g_\Lambda a^3 - \frac{g_r}{a} \right. \\ & \left. - \frac{g_s}{a^3} + \frac{1}{N^2} F(\phi) a^3 \dot{\phi}^2 \right), \end{aligned} \quad (13)$$

in which the new coefficients are defined as [31]

$$\begin{aligned} g_c &= \frac{6k}{3(3\lambda - 1)}, \\ g_\Lambda &= \frac{2\Lambda}{3(3\lambda - 1)}, \\ g_r &= \frac{12k^2(3g_2 + g_3)}{3(3\lambda - 1)M_{\text{PL}}^2}, \\ g_s &= \frac{24k^3(9g_4 + 3g_5 + g_6)}{3(3\lambda - 1)M_{\text{PL}}^4}. \end{aligned} \quad (14)$$

The momentum conjugate to each of the dynamical variables can be obtained by definition  $p_q = \partial L / \partial \dot{q}$  with results  $p_a = -(2a\dot{a}/N)$  and  $p_\phi = (2/N)F(\phi)a^3\dot{\phi}$ . In terms of these momenta, the total Hamiltonian reads

$$H = N\mathcal{H} = N \left( -\frac{p_a^2}{4a} + \frac{p_\phi^2}{4a^3 F(\phi)} - g_c a + g_\Lambda a^3 + \frac{g_r}{a} + \frac{g_s}{a^3} \right). \quad (15)$$

As it should be, the lapse function appears as a Lagrange multiplier in the Hamiltonian which means that the Hamiltonian equation for this variable yields the constraint equation  $\mathcal{H} = 0$ . At classical level, this constraint is equivalent to the Friedmann equation. As we shall see later, this constraint also plays a key role in forming the basic equation of quantum cosmology, that is, the WDW equation.

### 3. Classical Cosmology

In this section, we intend to study the classical cosmological solutions of the HL model whose Hamiltonian is given by equation (15). In the Hamiltonian approach, the classical dynamics of each variable is determined by the Hamilton equation  $\dot{q} = \{q, H\}$ , where  $\{.,.\}$  is the Poisson bracket. So, we get

$$\begin{cases} \dot{a} = -\frac{Np_a}{2a}, \\ \dot{p}_a = N \left( -\frac{p_a^2}{4a^2} + g_c - 3g_\Lambda a^2 + \frac{g_r}{a^2} + \frac{3g_s}{a^4} + \frac{3p_\phi^2}{4a^4 F(\phi)} \right), \\ \dot{\phi} = \frac{Np_\phi}{2a^3 F(\phi)}, \\ \dot{p}_\phi = \frac{Np_\phi^2 F'}{4a^3 F^2}, \end{cases} \quad (16)$$

where  $F' = dF(\phi)/d\phi$ . Before attempting to solve the above system of equations, we must choose a time gauge by fixing the Lapse function. Without this, we will face with the problem of underdeterminacy which means that there are fewer equations than unknowns. So, let us fix the lapse function as  $N = a^n(t)$ , where  $n$  is a constant. With this time gauge, by eliminating  $p_\phi$  from two last equations of (16), we obtain the following equation for  $\phi$ :

$$\frac{\ddot{\phi}}{\dot{\phi}} + \frac{1}{2} \frac{F'(\phi)}{F(\phi)} \dot{\phi} + (3-n) \frac{\dot{a}}{a} = 0, \quad (17)$$

which seems to be a conservation law for the scalar field. This equation can easily be integrated with result

$$\dot{\phi}^2 F(\phi) = Ca^{2(n-3)}, \quad (18)$$

where  $C$  is an integration constant. Now, to obtain a differential equation for the scale factor, let us eliminate the momenta from the system (16) from which and also using (18), we will arrive at

$$\dot{a}^2 + a^{2n} \left( g_c - g_\Lambda a^2 - \frac{g_r}{a^2} - \frac{g_s + C}{a^4} \right) = 0, \quad (19)$$

which is equivalent to the Hamiltonian constraint  $H = 0$ . The last differential equation we want to derive from the above relations is an equation between  $a$  and  $\phi$  whose solutions give us the classical trajectories in the plane  $a - \phi$ . This may be done by removing the time parameter between (18) and (19) which yields

$$F(\phi) \left( \frac{d\phi}{da} \right)^2 = Ca^{-6} \left( -g_c + g_\Lambda a^2 + \frac{g_r}{a^2} + \frac{g_s + C}{a^4} \right)^{-1}. \quad (20)$$

The nondependence of this equation on the parameter  $n$  indicates that although different time gauges lead to different functions for the scale factor and scalar field, the classical trajectories are independent of these gauges. On the other hand, from now on, to make the model simple and solvable, we take a polynomial coupling function for the scalar field as  $F(\phi) = \lambda\phi^m$ .

In general, the above equations do not seem to have analytical solutions, so in what follows, we restrict ourselves to some special cases for which we can obtain analytical closed form solutions for the above field equations.

*3.1. Flat Universe with Cosmological Constant:  $k = 0$ ,  $\Lambda \neq 0$ .* For a flat universe, the coefficients  $g_c$ ,  $g_r$ , and  $g_s$  vanish. So, if we choose the time parameter corresponding to the gauge  $N = 1$  or equivalently  $n = 0$ , the Friedmann equation (19) reads

$$\dot{a}^2 = g_\Lambda a^2 + \frac{C}{a^4}, \quad (21)$$

with solution

$$a(t) = \left( \frac{C}{g_\Lambda} \right)^{1/6} \sinh^{1/3}(3\sqrt{g_\Lambda}t), \quad (22)$$

where the integration constant is adopted in such a way that the singularity occurs at  $t = 0$ , which means that  $a(t = 0) = 0$ . Now, let us find an expression for the scalar field. As we mentioned before, we consider its self-coupling function in the form of  $F(\phi) = \lambda\phi^m$ . With this choice for the function  $F(\phi)$ , equation (18) takes the form

$$\phi^{m/2} d\phi = \pm \sqrt{\frac{C}{\lambda}} \frac{dt}{a^3(t)}, \quad (23)$$

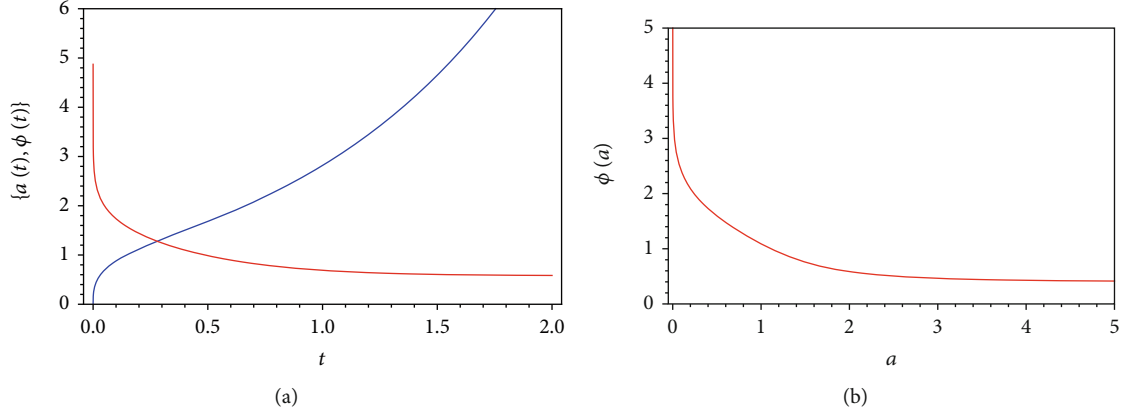


FIGURE 1: (a) The qualitative dynamical behavior of the scale factor (blue line) and scalar field (red line) in the flat universe. (b) Classical trajectory in the plane  $a - \phi$ . The figures are plotted for numerical values:  $g_\Lambda = 1$ ,  $C = 5$ ,  $\lambda = 1$ ,  $m = 2$ , and  $\phi_c = 2/5$ .

in which with the help of equation (22), we are able to integrate it and find the time evolution of the scalar field as

$$\phi(t) = \left[ \phi_c^{(m+2)/2} - \frac{m+2}{6\sqrt{\lambda}} \ln \left( \tanh \frac{3\sqrt{g_\Lambda} t}{2} \right) \right]^{2/(m+2)}, \quad m \neq -2, \quad (24)$$

where the integration constant  $\phi_c$  is chosen such that  $\lim_{t \rightarrow \infty} \phi(t) = \phi_c$ . In Figure 1, we have plotted the behavior of the scale factor and the scalar field. As this figure shows, the universe begins its evolution with a Big Bang singularity (zero size for  $a(t)$ ) at  $t=0$  where the scalar field blows up there. As time goes on, while the universe expands (with positive acceleration) until it finally enters to a de Sitter phase, that is, we have  $a(t) \sim e^{\sqrt{g_\Lambda} t}$ , as  $t \rightarrow +\infty$ , the scalar field eventually tends to have a constant value. We can also follow this behavior by studying the classical trajectories in the plane  $a - \phi$ . To do this, we need to extract the scalar field in terms of the scale factor from equation (20) which now can be written as

$$\phi^m \left( \frac{d\phi}{da} \right)^2 = \frac{C}{\lambda a^2 (g_\Lambda a^6 + C)}, \quad (25)$$

whose integration yields

$$\phi(a) = \left[ \phi_c^{(m+2)/2} + \frac{m+2}{6\sqrt{\lambda}} \ln \frac{\sqrt{g_\Lambda a^6 + C} - \sqrt{C}}{\sqrt{g_\Lambda} a^3} \right]^{2/(m+2)}, \quad m \neq -2. \quad (26)$$

In Figure 1, we also plotted the above expression for typical numerical values of the parameters. How the scale factor varies with respect to the scalar field, or vice versa, can also be seen from this figure. We will return to this classical trajectory again when looking at the quantum model to investigate whether the peaks of the wave function correspond to these paths.

**3.2. Nonflat Universe with Zero Cosmological Constant:  $k \neq 0$ ,  $\Lambda = 0$ .** In this section, we consider another special case in which while the curvature index is nonzero, the cosmological constant is equal to zero. This means that  $g_c, g_s, g_r \neq 0$  and  $g_\Lambda = 0$ . Under these conditions, if we take an evolution parameter corresponding to the lapse function  $N(t) = a(t)$  (or  $n = 1$ ), the Friedmann equation (19) will be

$$\dot{a}^2 + g_c a^2 - g_r - \frac{g_s + C}{a^2} = 0. \quad (27)$$

Before trying to solve this equation, we should note a point about the selected lapse function. Unlike the case in the previous section in which our time parameter with  $N = 1$  was indeed the usual cosmic time, here with  $N = a(t)$ ,  $t$  is just an evolution or clock parameter in terms of which the evolution of all dynamical variables is measured. However, one may translate the final results in terms of the cosmic time  $\tau$ , using its relation with the time parameter  $t$ , that is,  $d\tau = N(t)dt$ .

The general solution to equation (27) is

$$a(t) = \sqrt{\alpha(1 - \cos \omega t) + \beta \sin \omega t}, \quad (28)$$

where  $\alpha = g_r/2g_c$ ,  $\beta = \sqrt{(g_s + C)/g_c}$ , and  $\omega = 2\sqrt{g_c}$ . To express this and the following relations in a simpler form, let us take  $g_r = 0$ , which is equivalent to  $g_3 = -3g_2$  in (14). Also, we assume that  $\lambda > 1/3$  and  $9g_4 + 3g_5 + g_6 > 0$ , so that  $\text{sign}(g_c, g_s) = \text{sign}(k)$ . In the following, we will present the solutions for the closed universe  $k = +1$ . The open ( $k = -1$ ) counterpart of the solutions can be obtained via making small changes by replacing the trigonometric functions with their hyperbolic counterparts. Therefore, by applying, again, the initial condition  $a(t=0) = 0$ , we have

$$a(t) = \left( \frac{g_s + C}{g_c} \right)^{1/4} \sqrt{\sin(2\sqrt{g_c} t)}. \quad (29)$$

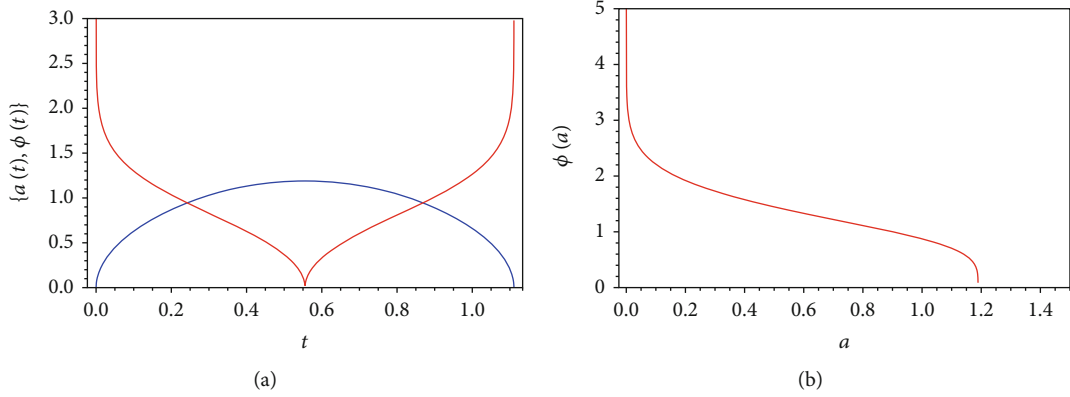


FIGURE 2: (a) The qualitative behavior of the scale factor (blue line) and scalar field (red line) when  $\Lambda=0$  and  $g_r=0$ . (b) The classical trajectory in the plane  $a - \phi$ . The figures are plotted for the numerical values:  $g_c = 2$ ,  $g_s = 1$ ,  $C = 3$ ,  $\lambda = 1$ ,  $m = 2$ , and  $\phi_c = 0$ .

The time dependence of the scalar field can also be deduced from equation (18) which for the present case has the solution

$$\phi(t) = \left[ \phi_c^{(m+2)/2} + \frac{m+2}{4} \sqrt{\frac{C}{(g_s+C)\lambda}} \operatorname{Intan}(\sqrt{g_c} t) \right]^{2/(m+2)}, \quad m \neq -2. \quad (30)$$

Finally, what remains is the classical trajectories in the plane  $a - \phi$  which as before may be obtained from (20) with result

$$\phi(a) = \left[ \frac{m+2}{4} \sqrt{\frac{C}{(g_s+C)\lambda}} \ln \frac{\sqrt{g_s+C} + \sqrt{g_s+C - g_c a^4}}{\sqrt{g_c} a^2} \right]^{2/(m+2)}. \quad (31)$$

In Figure 2, we have shown the time behavior of the scale factor and the scalar field. As clearly shown from this figure, the universe begins its decelerated expansion from a singularity where the size of the universe is zero and the value of the scalar field is infinite. As the scale factor expands to its maximum value, the scalar field decreases to zero. Then, by recollapsing the scale factor to a Big Crunch singularity, the scalar field again increases until it blows where the scale factor vanishes. The behavior of the scale factor and the scalar field in the plane  $a - \phi$  is also plotted in this figure. This figure also clearly shows the divergent behavior of the scalar field where the scale factor is singular.

**3.3. Early Universe.** In this section, we consider the dynamics of the universe in the early times of cosmic evolution when the scale factor is very small. For such a situation, the Friedmann equation (19) (again in the gauge  $N = a(t)$ ) takes the form

$$\dot{a}^2 = g_r + \frac{g_s + C}{a^2}, \quad (32)$$

in which we omit the terms containing  $a^2$  and  $a^4$ . It is easy to derive the scale factor from this equation as

$$a(t) = \left[ g_r(t + \delta)^2 - \frac{g_s + C}{g_r} \right]^{1/2}, \quad (33)$$

where  $\delta = \sqrt{g_s + C}/g_r$ . This equation shows that, regardless of whether the curvature index is positive or negative, the universe has a power law expansion in the early times of its evolution coming from a Big Bang singularity. Following the same steps we took in the previous sections will lead us to the following expressions for the scalar field and the classical trajectory:

$$\phi(t) = \left[ \phi_c^{(m+2)/2} - \frac{m+2}{4} \sqrt{\frac{C}{(g_s+C)\lambda}} \cdot \ln \frac{g_r(t + \delta) - \sqrt{g_s+C}}{g_r(t + \delta) + \sqrt{g_s+C}} \right]^{2/(m+2)}, \quad m \neq -2, \quad (34)$$

$$\phi(a) = \left[ \phi_c^{(m+2)/2} + \frac{m+2}{2} \sqrt{\frac{C}{(g_s+C)\lambda}} \cdot \ln \frac{\sqrt{g_r a^2 + g_s + C} + \sqrt{g_s + C}}{\sqrt{g_r} a} \right]^{2/(m+2)}. \quad (35)$$

As before, we summarized all the results of this section in Figure 3, which illustrates the evolution and singularity of the dynamical variables.

**3.4. Late Time Expansion.** Another important issue in cosmological dynamics is the late time behavior of the universe. In this limit, the Friedmann equation (19), in the gauge  $N = 1$ , has the form

$$\dot{a}^2 + g_c - g_\Lambda a^2 = 0, \quad (36)$$

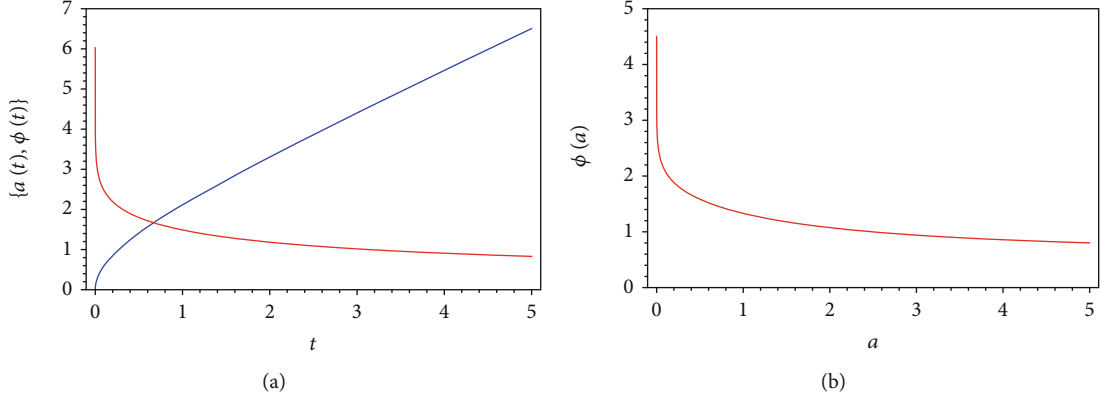


FIGURE 3: (a) The behavior of the scale factor (blue line) and scalar field (red line) in the early times of cosmic evolution. (b) The corresponding classical trajectory in the plane  $a - \phi$ . The figures are plotted for numerical values:  $g_r = 1$ ,  $g_s = 2$ ,  $C = 1$ ,  $\lambda = 1$ ,  $m = 2$ , and  $\phi_c = 1/2$ .

where here we have neglected the terms  $1/a^2$  and  $1/a^4$ . It is easy to see that this equation is solved by the following functions:

$$\begin{aligned} a(t) &= \frac{1}{2g_\Lambda} (e^{\sqrt{g_\Lambda}t} + g_c g_\Lambda e^{-\sqrt{g_\Lambda}t}), \\ a(t) &= \frac{1}{2g_\Lambda} (g_c g_\Lambda e^{\sqrt{g_\Lambda}t} + e^{-\sqrt{g_\Lambda}t}), \end{aligned} \quad (37)$$

of which both enter the de Sitter phase:

$$a(t) \sim e^{\sqrt{g_\Lambda}t}, \quad (38)$$

when  $t \rightarrow +\infty$ . Similar to the calculations of the preceding sections, we can obtain the following expression for the scalar field:

$$\begin{aligned} \phi(t) &= \left[ \phi_c^{(m+2)/2} + \frac{(m+2)g_\Lambda}{4g_c} \sqrt{\frac{C}{\lambda g_c}} \left( \frac{\tanh(\sqrt{g_\Lambda}t)}{\cosh(\sqrt{g_\Lambda}t)} \right) \right. \\ &\quad \left. + 2 \arctan(e^{\sqrt{g_\Lambda}t}) \right]^{2/(m+2)}, \end{aligned} \quad (39)$$

which tends to have a constant value as  $t \rightarrow +\infty$ .

#### 4. Canonical Quantization of the Model

As we mentioned before, HL gravity is a generalization of the usual GR at UV regimes in such a way that in the low-energy limits, the standard GR is recovered. Therefore, from a cosmological point of view, one may obtain some nonsingular bouncing solutions. From this perspective, this theory may be considered an alternative to inflation since it is expected that it might solve the flatness and horizon problems and generate scale invariant perturbations for the early universe without the need of exponential expansion usually used in the inflationary theories [32–34].

On the other hand, at the background (nonperturbation) level, almost all solutions to the Einstein field equations

exhibit different kinds of singularities. On this basis, cosmological solutions along with conventional matter fields are no exception to this rule and mainly exhibit Big Bang-type singularities. Any hope to eliminate these singularities would be in the development of a concomitant and conducive quantum theory of gravity. In the absence of a complete theory of quantum gravity, it would be useful to describe the quantum state of the universe in the context of quantum cosmology, in which based on the canonical quantization procedure, the evolution of the universe is described by a wave function in the minisuperspace. In other words, in this view, the system in question will be reduced to a conventional quantum mechanical system. In what follows, according to this procedure, we are going to overcome the singularities that appear in the classical model.

Now let us focus our attention on the quantization of the model described in the previous section. To do this, we start with the Hamiltonian (15). As we know, the lapse function in such Hamiltonians appears itself as a Lagrange multiplier, so we have the Hamiltonian constraint  $H = 0$ . This means that application of the canonical quantization procedure demands that the quantum states of the system (here the universe) should be annihilated by the quantum (operator) version of  $H$ , which yields the WDW equation  $\hat{H}\Psi(a, \phi) = 0$ , where  $\Psi(a, \phi)$  is the wave function of the universe. To obtain the differential form of this equation, if we use the usual representation  $P_q \rightarrow \partial_q$ , we are led to the following WDW equation:

$$\begin{aligned} &\frac{1}{4a} \left( \frac{\partial^2}{\partial a^2} + \frac{\beta}{a} \frac{\partial}{\partial a} \right) \Psi(a, \phi) \\ &- \frac{1}{4a^3 F(\phi)} \left( \frac{\partial^2}{\partial \phi^2} + \frac{\kappa F'(\phi)}{F(\phi)} \frac{\partial}{\partial \phi} \right) \Psi(a, \phi) \\ &+ \left( -g_c a + g_\Lambda a^3 + \frac{g_r}{a} + \frac{g_s}{a^3} \right) \Psi(a, \phi) = 0, \end{aligned} \quad (40)$$

where the parameters  $\beta$  and  $\kappa$  represent the ambiguity in the ordering of factors  $(a, P_a)$  and  $(\phi, P_\phi)$ , respectively. It is clear that there are lots of the possibilities to choose these

parameters. For example, with  $\beta = \kappa = 0$ , we have no factor ordering, and with  $\beta = \kappa = 1$ , the kinetic term of the Hamiltonian takes the form of the Laplacian  $-(1/2)\nabla^2$  of the min-superspace. In general, as it is clear from the WDW equation, the resulting form of the wave function depends on the chosen factor ordering [35]. However, it can be shown that the factor ordering parameter will not affect semiclassical calculations in quantum cosmology [36], and so for convenience, one usually chooses a special value for it in the special models.

As the first step in solving equation (40), let us separate the variables into the form  $\Psi(a, \phi) = \psi(a)\Phi(\phi)$ , which yields the following differential equations for the functions  $\psi(a)$  and  $\Phi(\phi)$ :

$$\frac{d^2\psi(a)}{da^2} + \frac{\beta}{a} \frac{d\psi(a)}{da} + 4 \left( -g_c a^2 + g_\Lambda a^4 + g_r + \frac{g_s + w^2}{a^2} \right) \psi(a) = 0, \quad (41)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \frac{\kappa F'(\phi)}{F(\phi)} \frac{d\Phi(\phi)}{d\phi} + 4w^2 F(\phi)\Phi(\phi) = 0, \quad (42)$$

with  $w$  being a separation constant. As in the classical case, here we examine the analytical solutions of the above equations in a few specific cases. Moreover, we assume that the wave functions are supposed to obey the boundary conditions:

$$\Psi(a=0, \phi) = 0, \text{ Dirichlet B.C.}, \quad (43)$$

$$\left. \frac{\partial \Psi(a, \phi)}{\partial a} \right|_{a=0} = 0, \text{ Neumann B.C.}, \quad (44)$$

where the first condition is called the Dewitt boundary condition to avoid the singularity in the quantum domain. In what follows, we will deal with the resulting quantum cosmology in the same special cases that we have already examined the classical solutions.

**4.1. Flat Quantum Universe with Cosmological Constant:  $\kappa = 0$ ,  $\Lambda \neq 0$ .** In this case by selecting  $\beta = -2$ , equation (41) reads as

$$\frac{d^2\psi(a)}{da^2} - \frac{2}{a} \frac{d\psi(a)}{da} + 4 \left( g_\Lambda a^4 + \frac{w^2}{a^2} \right) \psi(a) = 0, \quad (45)$$

of which the solutions in terms of Bessel functions  $J_\nu(z)$  and  $Y_\nu(z)$  are as follows:

$$\begin{aligned} \psi(a) = & C_1 a^{3/2} J_{\frac{i}{6}\sqrt{16w^2-9}} \left( \frac{2}{3} \sqrt{g_\Lambda} a^3 \right) \\ & + C_2 a^{3/2} Y_{\frac{i}{6}\sqrt{16w^2-9}} \left( \frac{2}{3} \sqrt{g_\Lambda} a^3 \right), \end{aligned} \quad (46)$$

where  $C_1$  and  $C_2$  are the integration constants. In the case where the order of Bessel functions is imaginary, ( $w^2 >$

$9/16$ ), both of them satisfy the DeWitt boundary conditions and so both integral constants can be nonzero which we take them here as  $C_1 = 1$  and  $C_2 = i$ .

On the other hand, by putting the coupling function  $F(\phi) = \lambda\phi^m$  and setting the ordering parameter as  $\kappa = 0$ , equation (42) takes the form

$$\frac{d^2\Phi(\phi)}{d\phi^2} + 4w^2 \lambda \phi^m \Phi(\phi) = 0, \quad (47)$$

with solutions

$$\begin{aligned} \Phi(\phi) = & C_3 \sqrt{\phi} J_{1/(m+2)} \left( \frac{4\sqrt{\lambda} w}{m+2} \phi^{(m+2)/2} \right) \\ & + C_4 \sqrt{\phi} Y_{1/(m+2)} \left( \frac{4\sqrt{\lambda} w}{m+2} \phi^{(m+2)/2} \right). \end{aligned} \quad (48)$$

Thus, the eigenfunctions of the WDW equation can be written as

$$\begin{aligned} \Psi_w(a, \phi) = & \psi_w(a)\Phi_w(\phi) = a^{3/2} \sqrt{\phi} H_{\frac{i}{6}\sqrt{16w^2-9}}^{(1)} \\ & \cdot \left( \frac{2}{3} \sqrt{g_\Lambda} a^3 \right) J_{1/(m+2)} \left( \frac{4\sqrt{\lambda} w}{m+2} \phi^{(m+2)/2} \right), \end{aligned} \quad (49)$$

where we choose  $C_4 = 0$ , for having well-defined functions in all ranges of variable  $\phi$ , and  $H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z)$  are the Hankel functions. We may now write the general solutions to the DWD equation as a superposition of the above eigenfunctions, that is,

$$\begin{aligned} \Psi(a, \phi) = & \int_D dw f(w) \Psi_w(a, \phi) \\ = & a^{3/2} \sqrt{\phi} \int_D dw f(w) H_{\frac{i}{6}\sqrt{16w^2-9}}^{(1)} \\ & \cdot \left( \frac{2}{3} \sqrt{g_\Lambda} a^3 \right) J_{1/(m+2)} \left( \frac{4\sqrt{\lambda} w}{m+2} \phi^{(m+2)/2} \right), \end{aligned} \quad (50)$$

where  $f(w)$  is a suitable weight function to construct the quantum wave packets and  $D = (-\infty, -3/4] \cup [3/4, +\infty)$  is the domain on which the integral is taken. It is seen that this expression is too complicated for extracting an analytical closed form for the wave function and the choice of a function  $f(w)$  that leads to an analytical solution for the wave function is not an easy task. However, such weight functions in quantum systems can be chosen as a shifted Gaussian weight function:

$$f(w) = w^p e^{-\sigma(w-w_0)^2}, \quad (51)$$

which are widely used in quantum mechanics as a way to construct the localized states. This is because these types of



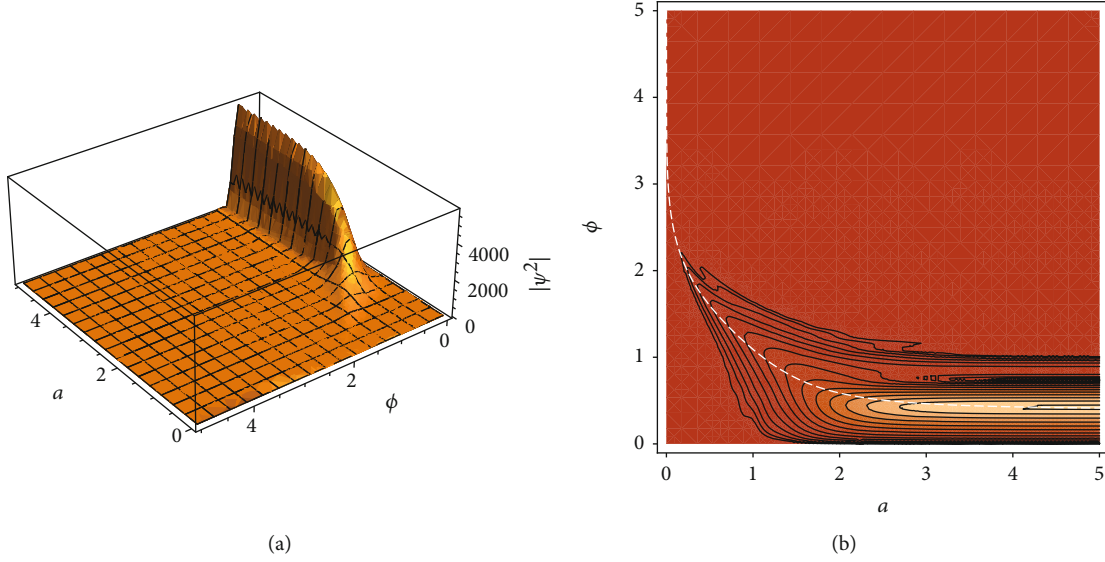


FIGURE 4: The figures show the square of the wave function (a) and its corresponding contour plot (b). Also, the dashed line denotes the classical trajectory of the system in the phase space. The figures are plotted for the numerical values:  $g_\Lambda = 1$ ,  $\lambda = 1$ ,  $m = 2$ ,  $p = 0$ ,  $\sigma = 1/10$ , and  $w_0 = 1/2$ .

weight factors are centered about a special value of their argument and they fall off rapidly away from that center. Due to this behavior, the corresponding wave packet resulting from (50) after integration has also a Gaussian-like behavior, i.e., localized about some special values of its arguments.

To realize the correlation between these quantum patterns and the classical trajectories, note that in the minisuperspace formulation, the cosmic evolution of the universe is modeled with the motion of a point particle in a space with minisuperspace coordinates. In this sense, one of the most important features in quantum cosmology is the recovery of classical solutions from the corresponding quantum model or, in other words, how can the WDD wave functions predict a classical universe. In quantum cosmology, one usually constructs a coherent wave packet with suitable asymptotic behavior in the minisuperspace, peaking in the vicinity of the classical trajectory. In Figure 4, we have plotted the qualitative behavior of the square of the wave function (50) with the above mentioned Gaussian weight factor and its contour plot for typical numerical values of the parameters. As this figure shows, while the wave function has its dominant peaks in the vicinity of the classical trajectories, these peaks predict a universe to come out of a nonzero value of the scale factor. Therefore, it can be seen that by the quantization of the model, while we are able to eliminate the classical singularity, we are also led to a quantum pattern with a good agreement with its classical counterpart.

**4.2. Nonflat Quantum Universe with Zero Cosmological Constant:  $k \neq 0$ ,  $\Lambda = 0$ .** In this case (equation (41)), the WDW equation for the scale factor takes the form

$$\frac{d^2\psi(a)}{da^2} + 4\left(-g_c a^2 + g_r + \frac{g_s + w^2}{a^2}\right)\psi(a) = 0, \quad (52)$$

in which we have set the ordering parameter as  $\beta = 0$ . The general solutions to this differential equation are

$$\psi(a) = C_5 \frac{M_{\mu\nu}(2\sqrt{g_c}a^2)}{\sqrt{a}} + C_6 \frac{W_{\mu\nu}(2\sqrt{g_c}a^2)}{\sqrt{a}}, \quad (53)$$

where  $M_{\mu\nu}(x)$  and  $W_{\mu\nu}(x)$  are Whittaker functions with  $\mu = g_r/2\sqrt{g_c}$  and  $\nu = (i/4)\sqrt{16(g_s + w^2) - 1}$ . If as in the classical case, we take  $g_r = 0$  (or equivalently  $g_3 = -2g_2$ ), the Whittaker functions can be expressed in terms of the modified Bessel functions  $K_{i\nu}(z)$  and  $I_{i\nu}(z)$  as follows [37]:

$$\psi(a) = C_7 \sqrt{a} K_{\frac{i}{4}\sqrt{16(g_s + w^2) - 1}}(\sqrt{g_c}a^2) + C_8 \sqrt{a} I_{\frac{i}{4}\sqrt{16(g_s + w^2) - 1}}(\sqrt{g_c}a^2). \quad (54)$$

Since the wave functions must satisfy  $\lim_{a \rightarrow \infty} \psi(a) = 0$ , we restrict ourselves to consider only the modified Bessel function  $K_{i\nu}(z)$  as solution and so we set  $C_8 = 0$ . The other part of the WDW equation is the equation of the scalar field which in this case is also the same as equation (47), and its solutions have already been given in relation (48). Therefore, if the coefficients are selected as  $C_3 = 1$  and  $C_4 = i$ , the eigenfunctions of the WDW equation read

$$\Psi_w(a, \phi) = \sqrt{a\phi} K_{i/4\sqrt{16(g_s + w^2) - 1}}(\sqrt{g_c}a^2) \cdot H_{1/(m+2)}^{(1)}\left(\frac{4\sqrt{\lambda}w}{m+2}\phi^{(m+2)/2}\right), \quad (55)$$

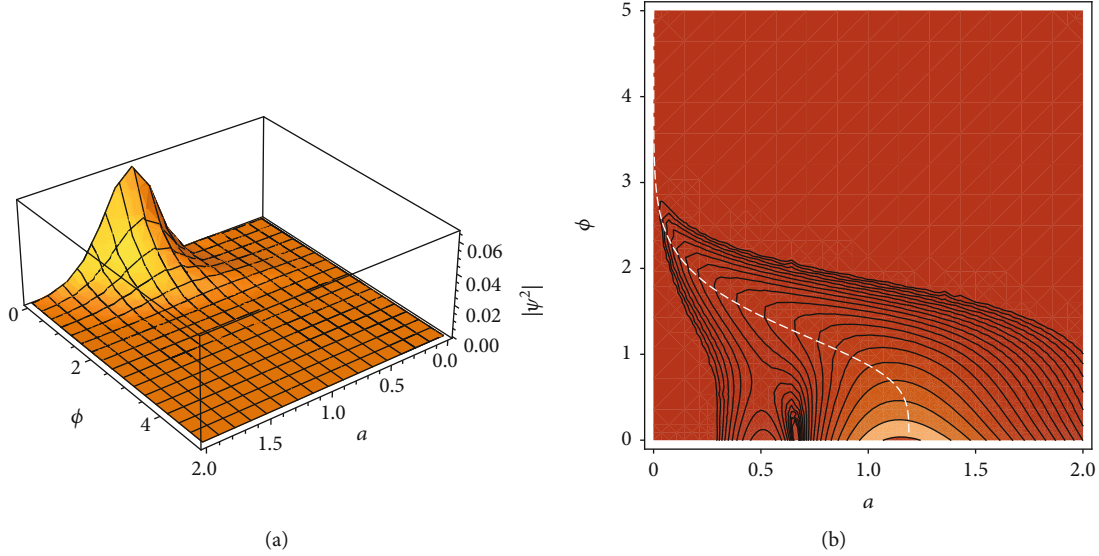


FIGURE 5: The figures show the square of the wave function (a) and its corresponding contour plot (b). Also, the dashed line denotes the classical trajectory of the system in the plane  $a - \phi$ . The figures are plotted for the numerical values:  $g_c = 2$ ,  $g_s = 1$ ,  $\lambda = 1$ ,  $m = 2$ ,  $p = 2$ ,  $\sigma = 1/2$ , and  $w_0 = 3$ .

in which their superposition gives the total wave function as

$$\Psi(a, \phi) = \sqrt{a\phi} \int_D 'dw f(w) K_{\frac{i}{4}\sqrt{16(g_s+w^2)-1}} \cdot (\sqrt{g_c} a^2) H_{1/(m+2)}^{(1)} \left( \frac{4\sqrt{\lambda} w}{m+2} \phi^{(m+2)/2} \right), \quad (56)$$

where  $f(w)$  is again a Gaussian-like weight factor in the form (51) and  $D'$  is the domain of integration over  $w$  as

$$D' = \begin{cases} (-\infty, +\infty), & g_s \geq \frac{1}{16}, \\ \left( -\infty, -\frac{1}{4}\sqrt{1-16g_s} \right) \cup \left( \frac{1}{4}\sqrt{1-16g_s}, +\infty \right), & g_s < \frac{1}{16}. \end{cases} \quad (57)$$

The results of the numerical study of this wave function are shown in Figure 5. The similarities and differences between quantum and classical solutions, and the fact that the wave functions' peaks correspond very well to the classical trajectories, are similar to those described at the end of the previous section.

**4.3. Early Quantum Universe.** In this last part of the article, we study the quantum dynamics of the model in the early times of evolution of the universe. As it is well known, the quantum behavior of the universe is more important in this period, and it is the quantum effects in this era that prevent the classical singularities. For small values of the scale factor, equation (41) with  $\beta = 0$  takes the form

$$\frac{d^2\psi(a)}{da^2} + 4 \left( g_r + \frac{g_s + w^2}{a^2} \right) \psi(a) = 0, \quad (58)$$

with solutions

$$\psi(a) = C_9 \sqrt{a} J_{(i/2)\sqrt{16(g_s+w^2)-1}}(2\sqrt{g_r}a) + C_{10} \sqrt{a} Y_{(i/2)\sqrt{16(g_s+w^2)-1}}(2\sqrt{g_r}a). \quad (59)$$

Since both Bessel functions satisfy the DeWitt boundary condition, both can contribute to making the wave function, so we take the coefficients as  $C_9 = 1$  and  $C_{10} = i$ . The solutions for the scalar field are the same as the ones we presented in the previous two sections. So, the final form of the wave function in this case is

$$\begin{aligned} \Psi(a, \phi) &= \int_D 'dw f(w) \psi_w(a) \Phi_w(\phi) \\ &= \sqrt{a\phi} \int_D 'dw f(w) H_{(i/2)\sqrt{16(g_s+w^2)-1}}^{(1)} \\ &\quad \cdot (2\sqrt{g_r}a) J_{1/(m+2)} \left( \frac{4\sqrt{\lambda} w}{m+2} \phi^{(m+2)/2} \right), \end{aligned} \quad (60)$$

where as before  $f(w)$  is a Gaussian-like weight function and the domain of integration  $D'$  is given by (57). The final results are shown in Figure 6. A look at this figure shows that the universe started its evolution from a nonzero value for the scale factor which in turn means that quantum effects have eliminated the singularity of the classical model. Also, as the figure clearly shows, the wave function has its peaks in the vicinity of classical trajectories shown in Figure 3, which indicate the compatibility of classical and quantum solutions.

## 5. Summary

In this paper, we have studied the classical and quantum FRW cosmology in the framework of the projectable HL

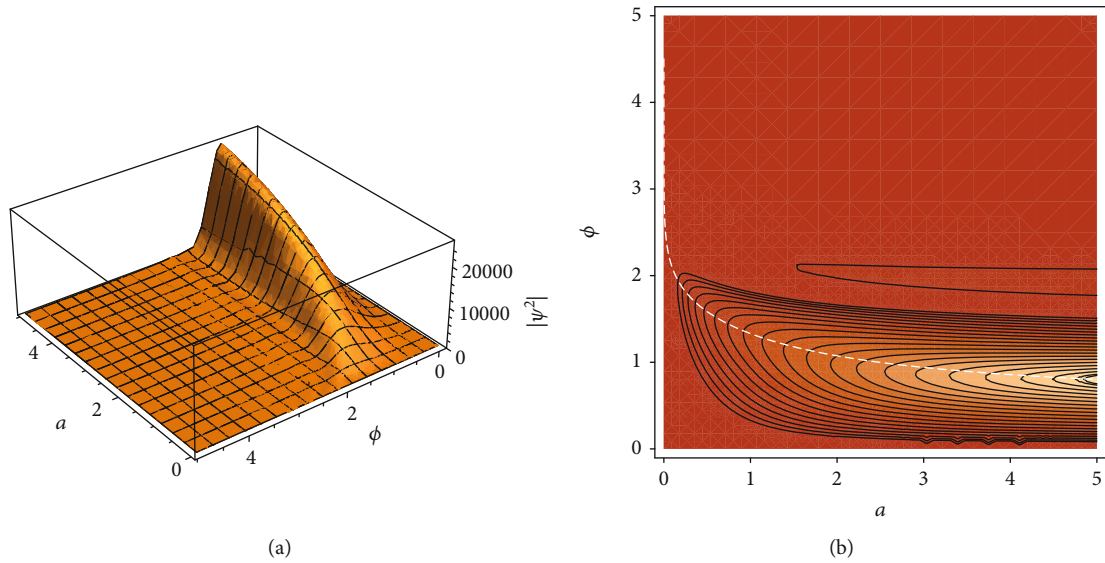


FIGURE 6: The figures show the square of the wave function (a) and its corresponding contour plot (b) for the early times of cosmic evolution. Also, the dashed line denotes the classical trajectory of the system in the plane  $a - \phi$ . The figures are plotted for the numerical values:  $g_r = 1$ ,  $g_s = 2$ ,  $\lambda = 1$ ,  $m = 2$ ,  $p = 0$ ,  $\sigma = 1$ , and  $\omega_0 = 1/2$ .

theory of gravity without the detailed balance condition. The phase space variables turn out to correspond to the scale factor of the FRW metric and a nonlinear self-coupling scalar field with which the action of the model is augmented. After an introduction to the HL theory, based on a dimensional analysis, we present the terms which are allowed to be included in the potential part of the action of this theory. This process enabled us to write the Lagrangian and Hamiltonian of the model in terms of the minisuperspace variables and some correction parameters coming from the HL theory.

We then studied the classical cosmology of this model and formulate the corresponding equations within the framework of the Hamiltonian formalism. Though, in general, the classical equations did not have exact solutions, we analyzed their behavior in the special cases of the flat universe with cosmological constant, the nonflat universe with vanishing cosmological constant, and the early and late times of cosmic evolution and obtained analytical expressions for the scale factor and the scalar field in each of these cases. Another point to note about the classical solutions is the choice of the appropriate lapse function in each case, which actually represents the time gauge in which that solution is obtained. We have seen that the classical expressions for the scale factor and scalar field exhibit some kinds of classical singularities. These singularities are mainly of the Big Bang type for the scale factor and blowup type for the scalar field.

The last part of the paper is devoted to the quantization of the model described above in which we saw that the classical singular behavior will be modified. In the quantum models, we separated the WDW equation and showed that its eigenfunctions can be obtained in terms of analytical functions. By an appropriate superposition of the eigenfunctions, we constructed the integral form of the wave functions. Although it is seen that these integral expressions are too complicated for extracting an analytical closed form for the wave functions, employing numerical methods, we have plotted the

approximate behavior of the square of the wave functions for typical values of the parameters. In each case, investigation of the pattern followed by the wave functions show a nonsingular behavior near the classical singularity. In addition to singularity avoidance, we saw that the wave functions' peaks are with a good approximation, in the vicinity of the classical trajectories, which indicates the fact that the classical and quantum solutions are in complete agreement with each other in the late time of cosmic evolution.

## Data Availability

No data were used to support the authors' study in this manuscript.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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