



Non-negative Integer Power of a Hyponormal m -isometry is Reflexive

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Authors' contributions

This work was carried out in collaboration among all authors. Author PK designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors RKP and DN managed the analyses of the study. Author DN managed the literature searches. All authors read and approved the final manuscript.

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Abstract

Sarason did pioneer work on reflexive operator and reflexivity of normal operators, however, he did not use the word reflexive but his results are equivalent to say that every normal operator is reflexive. The word reflexive was suggested by HALMOS and first appeared in H. Rajdavi and P. Rosenthals book 'Invariant Subspaces' in 1973. This line of research was continued by Deddens who showed that every isometry in $B(H)$ is reflexive. R. Wogen has proved that 'every quasi-normal operator is reflexive'. These results of Deddens, Sarason, Wogen are particular cases of theorem of Olin and Thomson which says that all sub-normal operators are reflexive. In other direction, Deddens and Fillmore characterized these operators acting on a finite dimensional space are reflexive. J. B. Conway and Dudziak generalized the result of reflexivity of normal, quasi-normal, sub-normal operators by proving the reflexivity of Vonneumann operators. In this paper we shall discuss the condition under which m -isometries operators turned to be reflexive.

Keywords: Reflexive operators; commutant property; isometry; m -isometric unilateral weighted shift.

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1 Introduction

A bounded linear operator T on a complex separable Hilbert space H is reflexive if $\text{Alg } T = \text{Alg Lat } T$, where $\text{Alg Lat } T$ and $\text{Alg } T$ denote respectively the weakly closed algebra of operators which leave invariant every invariant sub-space of T and the weakly closed algebra generated by T and I .

An operator T has double commutant property if $\{T\}'' = \text{Alg } T$.

J. Agler and M. Stankus introduced an m -isometry [1,2,3,4]. Let H be a complex Hilbert space and $B(H)$ be a set of all bounded linear operators on H . Let $\binom{m}{k}$ be the binomial coefficient.

An operator $T \in B(H)$ is said to be an m -isometry if $\sum_{k=0}^m (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0$. It is known that for $m=1$ it is an isometry ($m = 1$). m -Isometries are not only a natural extension of an isometry, but they also play a very important role in the study of Dirichlet operators and some other classes of operators [5,6,7].

Let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ denote spectrum, point spectrum, approximate point spectrum respectively, of T . Also D and ∂D represent, respectively, the open unit disc and its boundary.

Definition: m -isometric unilateral weighted shifts. An operator T is called a unilateral weighted shift, if there exists an orthonormal basis $\{e_n : n \geq 0\}$ and a sequence $\{w_n\}_{n=0}^\infty$ of bounded complex numbers such that $Te_n = w_n e_{n+1}$ for all $n \geq 0$. The iterates of T are given by $T^0 = I$, and for $k > 0$,

$$T^k e_n = \left(\prod_{i=0}^{k-1} w_{n+i} \right) e_{n+k} \quad (n \geq 0).$$

It is known that T is an isometry if and only if $w_n \in \partial D$, for all $n \geq 0$. [8,9,10] Characterized all

2-isometric unilateral weighted shifts that are not isometries in terms of their weight sequences. Recall that a unilateral weighted shift T is unitarily equivalent to a weighted shift operator with a non-negative weight sequence. So we can assume that $w_n \geq 0$ for every $n \geq 0$. Furthermore, if T is injective, it can be assumed that $w_n > 0$ for every $n \geq 0$ [11,12,13,14].

In [15] it was shown that contraction operators whose essential spectrum in D , $\sigma_e(T)$, the essential spectrum of T , is dominating for ∂D (i.e., almost every point of ∂D is a non-tangential limit point of $\sigma_e(T) \cap D$) are reflexive. Afterwards, in [16], the reflexivity of some contractions with rich spectrum was discussed. In fact, it was shown that, if T is a contraction on a Hilbert space H so that $I - T^*T$ is a trace class operator and $\sigma(T) = D$, then T is reflexive. Now a question arises that if T is a contraction so that $D \subseteq \sigma_p(T)$, is T necessarily reflexive? In this direction some developments have occurred in [17] by giving sufficient conditions for an arbitrary contraction to be reflexive. Especially, it is shown that if T is a contraction so that $\sigma(T)$ contains ∂D , then either T is reflexive or has a nontrivial hyper invariant subspace. The reflexivity of a contraction whose spectrum is the closed unit disc D and non-negative integer powers of hypo normal m -isometries will be established.

Lemma 1. Let T be an m -isometric unilateral weighted shift with weight sequence $\{w_n\}_{n=0}^\infty$ and put $f(n) = (-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{n+i}^2$

If $f(0) = 0$, then $f(n) = 0$ for all non-negative integers n .

Proof- We can prove the above lemma by using mathematical induction. The result is true for $n = 0$. Let it is true for $n = j$. Since T is an m -isometry

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0 \quad \forall x \in H$$

for $x = e_n$ states that, for every non-negative integer n ,

$$(-1)^{m-1} + \sum_{k=1}^m (-1)^{m-k-1} \binom{m}{k} \prod_{i=0}^{k-1} w_{n+i}^2 = 0$$

for $n = j$

$$\begin{aligned} & (-1)^{m-1} + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{j+i}^2 \right] + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k-1} \prod_{i=0}^{k-1} w_{j+i}^2 \right] - \binom{m-1}{k-1} \prod_{i=0}^{m-1} w_{j+i}^2 = 0 \\ & \{ (-1)^{m-1} + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{j+i}^2 \right] + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k-1} \prod_{i=0}^{k-1} w_{j+i}^2 \right] - \binom{m-1}{k-1} \prod_{i=0}^{m-1} w_{j+i}^2 = 0 \} \times w_j^2 \\ & (-1)^{m-2} + \left[\sum_{k=2}^m (-1)^{m-k-1} \binom{m-1}{k-1} \prod_{i=1}^{k-1} w_{j+i}^2 \right] = 0 \\ & (-1)^{m-1} + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{j+i+1}^2 \right] = 0 \end{aligned}$$

So the result is true for $n=j + 1$. Hence by the principal of mathematical induction the result will be true for any value of n .

Theorem-1[18]. Suppose that T is a unilateral weighted shift operator with weights $\{w_n\}_{n=0}^\infty$. Then T is an m -isometry which is not an $m-1$ -isometry if and only if the following hold for every nonnegative integer n

$$(-1)^m + \left[\sum_{k=1}^m (-1)^{m-k} \binom{m}{k} \prod_{i=0}^{k-1} w_{n+i}^2 \right] = 0$$

And

$$(-1)^{m-1} + \left[\sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{n+i}^2 \right] \neq 0$$

Athavale [2] showed that, in general, not all m -isometries are isometries but under some conditions m -isometries turned to be isometry.

Theorem 2. Let $T \in B(H)$ be a unilateral weighted shift which is an m -isometry. If for some non-zero $x \in H$, $\|x\| = \|Tx\| = \dots = \|T^{m-1}x\|$

then T is an isometry.

Proof. Suppose that $\{e_n\}_{n=0}^\infty$ is an orthonormal basis for H and $Te_n = w_n e_{n+1}$ for all $n \geq 0$. Put $x = \sum_{n=0}^\infty \beta_n e_n$. Then we may have

$$\begin{aligned} 0 &= \sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \|T^k x\|^2 = \sum_{n=0}^\infty |\beta_n|^2 \left[(-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{n+i}^2 \right] \\ &= \sum_{n=0}^\infty |\beta_n|^2 < \Delta_T e_n, e_n > . \end{aligned}$$

Since $x \neq 0$, $\beta_n \neq 0$ for some n_0 . So the positivity of Δ_T implies that

$$< \Delta_T e_{n_0}, e_{n_0} > = (-1)^{m-1} + \sum_{k=1}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \prod_{i=0}^{k-1} w_{n_0+i}^2 = 0$$

Thus, by the negation of condition in Theorem 1 T must be an $m-1$ -isometry. Now, using same technique as above and using Theorem 1, $m - 1$ times, T must be an isometry.

Theorem-3 [19]- If T is an injective unilateral weighted shift operator and T^* has a non-zero eigenvalue, then for every positive integer n , the operator T^n is reflexive

It is known that every contraction m -isometry is an isometry [20,21] and so is reflexive.

Let $T \in B(H)$ and K be a compact subset of \mathbb{C} . By $\|f\|_K$ we mean $\sup \{|f(x)| : x \in K\}$. The set K is said to be a spectral set for T if $\sigma(T) \subseteq K$ and $\|f(T)\| \leq \|f\|_K$ for every rational function f with poles off K . If $\sigma(T)$ is a spectral set, we say that T is a von Neumann operator. Subnormal operators are well-known examples of von Neumann operators [22]. Conway and Dudziak [23] have shown that every von Neumann operator is reflexive. Hyponormal operator T with $\sigma(T) = \{z : |z| \leq r(T)\}$, where $r(T)$ is the spectral radius of T , is a Von Neumann operator and so is reflexive.

Theorem 4- If $T \in B(H)$ is a contraction so that $\sigma(T) = D$, then T is a reflexive operator.

Proof- The hypotheses imply that $T = 1$. Let f be a rational function with poles off D . Choose $R > 0$ so that $D \subseteq D(0, R) := \{z : |z| < R\}$, and f is analytic on $D(0, R)$. Suppose that $\sum_{k=0}^{\infty} a_k z^k$ denotes the power series representation of f . Then the sequence of its partial sum, defined by $p_n(z) = \sum_{k=0}^n a_k z^k$ converges uniformly to f on compact subsets of $D(0, R)$. By Von Neumann's inequality [18, Problem 229], $p_n(T) \leq \sup\{|p_n(z)| : z \in D\}$. Thus, using Riesz functional calculus we get $f(T) \leq \sup\{|f(z)| : z \in D\}$. So T is a Von Neumann operator and so is reflexive.

Theorem 5- Every non-negative integer power of a hyponormal m -isometry T , is reflexive.

Proof. Since T is hyponormal, $r(T) = T$ and m -isometry, $\sigma(T) = D$ or ∂D , and so $r(T) = 1$. Hence, T is a contraction m -isometry, so is an isometry [20]. This, in turn, implies that T^n is reflexive for $n \geq 0$.

2 Conclusion

Now we may conclude that every hyponormal m -isometric operator with non-negative integer power is reflexive. Further an open question may be raised that if condition of 'non negative integer power' is dropped will it still be reflexive operator?

Competing Interests

Authors have declared that no competing interests exist.

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