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Solitary Wave Solutions for the Shallow Water Wave Equations and the Generalized Klein-Gordon Equation Using Exp$\left(-\phi(\eta)\right)$-Expansion Method

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Authors' contributions

This work was carried out in collaboration between both authors. Author MMR designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author WK managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this article, we investigate the exact and solitary wave solutions for the shallow water wave equations and the generalized Klein-Gordon equation using the $\exp\left(-\phi(\eta)\right)$-expansion method. A wave transformation is applied to convert the problem into the form of an ordinary differential equation. By using this method, we found the explicit solitary wave solutions in terms of the hyperbolic functions, trigonometric functions, exponential functions and rational functions. The extracted solution plays a significant role in many physical phenomena such as electromagnetic waves, nonlinear lattice waves, ion sound waves in plasma, nuclear physics, shallow water waves and so on. It is noted that the method is reliable, straightforward and an effective mathematical tool for analytic treatment of nonlinear systems of partial differential equation in mathematical physics and engineering.

Keywords: The $\exp\left(-\phi(\eta)\right)$-expansion method; shallow water wave equation; Klein-Gordon equation; solitary wave solution; traveling wave solutions.

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1 Introduction

Solitary wave solutions of nonlinear partial differential equations (PDEs) play an important role in the study of nonlinear physical phenomena in industry and nature. The solitary wave phenomena are observed in various fields, such as in plasma physics, fluid dynamics, optical fibres, the Bose-Einstein condensates, biological systems, propagation of shallow water waves, etc. [1,2]. The shallow water waves describe the motion of water bodies that are seen in various places like sea beaches, lakes and rivers, and governed by Boussinesq equation [2-4]. The Boussinesq-like equations appear in many physical applications, such as nonlinear lattice wave, solitons in plasma, shallow water waves, and nuclear physics all are governed by nonlinear waves equation [4,5]. The Korteweg-de Varies (KdV) equation, Boussinesq equation, Klein-Gordon equation, regularized long wave (RLW), Benjamin-Bona-Mahoney (BBM) equation, and Kadomtsev Petviashvili (KP) are well known models of shallow water waves [5-7]. These equations are used to model in many physical phenomena, such as the hydrodynamics of lakes, storm surges, tidal flats, coastal regions and tsunami waves, as well as deep ocean tides. Nevertheless, the Boussinesq equation and generalized Klein-Gordon (KG) equations gives much better approximation to such waves. Among the nonlinear wave equations, the Bussinesq equation describes the small amplitude with uniform depth regime for long waves transmitting on the surface of shallow water [6,7]. Biswas et al. [8] studied the soliton solution to the KG equation with the effect of power law nonlinearities is considered for this equation. These solutions will be useful in carrying out further analysis of shallow water waves that arises in the context of oceanography and atmospheric science as a paradigm for geophysical fluid motions.

There are various mathematical models have been employed for obtaining exact and solitary wave solutions of nonlinear engineering problems. Some of these well-known models are, as examples, the exp-functions method [9,10], the modified simple equation method [11], the Jacobi elliptic function expansion method [12,13], the Adomian decomposition method [14], the F-expansion method [15], the homogenous balance method [16], the (G'/G) -expansion method [17,18], the novel (G'/G)-expansion method [19-21], the new generalized (G'/G)-expansion method [22,23] and so on has been used to solve different types of nonlinear systems of partial differential equations (PDEs). Recently, the $\exp(-\phi(\eta))$-expansion method has become widely applied to construct for traveling wave solutions of nonlinear equations in science and engineering [24-27]. For example, this method has been utilized to construct traveling wave solutions of the Pochhammer-Chree equation by Nematollah et al. [28] and Rashid et al. [29] also have used this method for constructing traveling wave solutions of nonlinear evolution equations. Therefore, in this article, the $\exp(-\phi(\eta))$-expansion method will be applied for obtaining exact and soliton solutions of shallow water wave equations and the generalized Klein-Gordon equation.

The rest of this article is organized as follows. In section 2, the basic ideas of the $\exp(-\phi(\eta))$-expansion method are expressed. In section 3, the method is employed of obtaining the exact and soliton solutions of the system of shallow water wave equations and the nonlinear generalized Klein-Gordon equations. In section, 4, physical explanations and graphical representations of the solutions are presented. Finally, conclusions are summarized in the last section.

2 Outline of the Exp$(-\phi(\eta))$-Expansion Method

In this section, we illustrate the basic idea of the exp $(-\phi(\eta))$-expansion method for obtaining exact solutions of shallow water wave equation and the generalized Klein-Gordon equation.

Consider a general nonlinear partial differential equation with independent variables x and t of the form

$$\left.\begin{array}{l} P_1(u, v, u_t, u_x, v_x, u_{xx}, v_{xx}, u_{tt}, \dots\dots\dots\dots\dots) = 0 \\ P_2(u, v, u_t, u_x, v_x, u_{xx}, v_{xx}, u_{tt}, \dots\dots\dots\dots\dots) = 0 \end{array}\right\} \tag{1}$$

where $u = u(x, t)$ and $v = v(x, t$ is an unknown function, P_1 and P_2 are polynomials of the variables u and v and its partial derivatives in which highest order derivatives and nonlinear terms are involved. The main steps of this method are given in the following:

Step 1: Consider the traveling wave transformation variables

$$u(x, t) = u(\eta), v(x, t) = v(\eta), \quad \eta = x - ct, \tag{2}$$

where $u(\eta)$, and $v(\eta)$ represents the wave solutions and 'c' is the wave speed. We obtain the following relations:

$$\frac{\partial}{\partial t}(.) = -c\frac{d}{d\eta}(.), \quad \frac{\partial}{\partial x}(.) = \frac{d}{d\eta}(.), \ \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\eta^2}(.). \tag{3}$$

Substituting Eq. (3) along with Eq. (2) in Eq. (1), we reduce Eq. (1) to the following ordinary differential equation (ODE) for $u = u(\eta)$:

$$\left.\begin{array}{l} \Re_1\ (u, v\ u', v', u'', v'',\ u''', v''', \dots\dots\dots\dots\dots) = 0 \\ \Re_2\ (u, v\ u', v', u'', v'',\ u''', v''', \dots\dots\dots\dots\dots) = 0 \end{array}\right\}, \tag{4}$$

where $\Re_1$ and $\Re_2$ being another polynomials form of their argument. Here prime denotes the derivative with respect to η. Integrating Eq. (4), as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions.

Step 2: Assume the traveling wave solution for the Eq. (4) can be articulated as a finite series in $\phi(\eta)$ as follows:

$$\left.\begin{array}{l} u(x,t) = u(\eta) = \sum_{i=1}^{m} A_i \ \exp(-\phi(\eta)) \\ v(x,t) = v(\eta) = \sum_{i=1}^{n} A_i \ \exp(-\phi(\eta)) \end{array}\right\} \tag{5}$$

where the parameters $\boldsymbol{m}$, and $\boldsymbol{n}$ can be found by balancing the highest-order linear term with the nonlinear terms in Eq. (4) and $\mathbf{A_i}$ $(\mathbf{0} \leq \mathbf{i} \leq \mathbf{m}, \mathbf{n})$ are constants to be determined, such that $\mathbf{A_m} \neq \mathbf{0}, \mathbf{A_n} \neq \mathbf{0}$ and $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\eta})$ satisfies the following auxiliary equation:

$$\phi'(\eta) = exp\big(-\phi(\eta)\big) + \mu\ exp\big(\phi(\eta)\big) + \lambda. \tag{6}$$

Depending on the parameters involved, Eq. (6) has the following subsequent solutions:

Family 1: Hyperbolic function solution, when $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$\phi(\eta) = ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)}\ tanh\left(\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\ (\eta + E)\right) - \lambda}{2\mu}\right). \tag{7}$$

Family 2: Trigonometric function solutions, when $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$\phi(\eta) = ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\ \ tan\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\ (\eta+E)\right)-\lambda}{2\mu}\right). \tag{8}$$

Family 3: Exponential function solutions, when $\mu = 0,\ \lambda \neq 0,\ \ \lambda^2 - 4\mu > 0$,

$$\phi(\eta) = -\, ln\left(\frac{\lambda}{exp(\lambda(\eta+E))-1}\right). \tag{9}$$

Family 4: Rational function solutions, when $\mu \neq 0,\ \lambda \neq 0,\ \ \lambda^2 - 4\mu = 0$,

$$\phi(\eta) = ln\left(-\frac{2(\lambda(\eta+E)+2)}{\lambda^2(\eta+E)}\right). \tag{10}$$

Family 5: when $\mu = 0,\ \lambda = 0,\ \ \lambda^2 - 4\mu = 0$,

$$\phi(\eta) = ln\,(\eta + E). \tag{11}$$

Here E is an integrating constant and $A_m, c, \lambda,\ \mu$ are constants to be determined latter, $A_m \neq 0$.

Step 3: The positive integer m and n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms of the highest order appearing in Eq. (4).

Step 4: We substitute Eq. (5) with Eq. (6) in Eq. (4) and then we take into consideration the function the exp $(-\phi(\eta))$. In consequence of this substitution, we obtain a polynomial in exp $(-\phi(\eta))$. We collect all the coefficients of identical power of exp $(-\phi(\eta))$ and equalize to zero delivers a system of algebraic equations whichever can be solved to find $A_m, \ldots.. V, \lambda,\ \mu$. The values of $A_m, \ldots\ldots.. V, \lambda,\ \mu$ along with the general solutions of Eq. (6), we obtain traveling wave solutions $u(x,t)$ of the nonlinear evolution of Eq. (1). The exp $(-\phi(\eta))$-expansion method seems to be a powerful tool in dealing with nonlinear physical models.

3 Application of Exp$\left(-\phi(\eta)\right)$-Expansion Method to Nonlinear PDEs

In this section, we apply the of exp$\left(-\phi(\eta)\right)$-expansion method to construct the exact and solitary wave solutions for the shallow water wave equations and the generalized Klein-Gordon equation.

3.1 The shallow water wave equations

A well-known model of nonlinear dispersive shall water waves, which was first introduced by Joseph Vlentin Boussinesq is formulated as [3,30-34]

$$\begin{cases} u_t + (uv)_x + v_{xxx} = 0 \\ \ \ v_t + u_x + vv_x = 0 \end{cases}, \tag{12}$$

where, $u(x,t)$ is the elevation of the water wave above a horizontal bottom and $v(x,t)$ is the surface velocity of water along the x-direction that deviate from equilibrium position of water. Eq. (12) is also known modified Boussinesq equation [35,36] that describe the evolutions of the water-surface elevation and of the depth-averaged velocity of small amplitude waves with long wavelengths. Boussinesq-like equations also appear in many physical phenomena, such as electromagnetic waves in nonlinear dielectrics, one-dimensional nonlinear lattice waves, ion sound waves in plasma, and oscillations in a nonlinear string. Yan and Zhang [37] studied this equation and obtained solitary wave solutions via different transformation.

We introduce the transformation $\eta = x - ct$ where c is constant, $u(x,t) = u(\eta)$ and $v(x,t) = v(\eta)$, the nonlinear partial differential equation (PDE) Eq. (12) is transformed to the ODE:

$$\begin{cases} -cu' + vu' + uv' + v''' = 0 \\ u' - cv' + vv' = 0 \end{cases}. \tag{13}$$

Integrating once the second equation of Eq. (13) and setting the integration constant to zero yields:

$$u = cv - \frac{v^2}{2}. \tag{14}$$

Substituting Eq. (14) into the first equation of the system of Eq. (13) we obtain

$$v''' + \left(3cv - \frac{3v^2}{2} - c^2\right) v' = 0. \tag{15}$$

Integrating Eq. (15) and neglecting the constant of integration, we obtain

$$v'' + \frac{3}{2}cv^2 - \frac{1}{2}v^3 - c^2 v = 0, \tag{16}$$

To determine the index n, we balance the linear term of the highest order derivative with the highest order nonlinear terms. Therefore, taking Eq. (5) in Eq. (16) we balance v'' and v^3, so that $3n = n + 2$, and this gives us $n = 1$.

Therefore, the solution of Eq. (16) can be expressed by a polynomial in $exp\big(-\phi(\eta)\big)$ as follows:

$$v(\eta) = A_0 + A_1\big(exp\big(-\phi(\eta)\big)\big), \tag{17}$$

whereas $v(\eta)$ is a solution of Eq. (16) and A_0 and A_1 are constants to be determined later such that $A_n \neq 0$, while λ, μ are arbitrary constants. It is easy to see that

$$v'(\eta) = -A_1\, exp\big(-2\phi(\eta)\big) - A_1\lambda\, exp\big(-\phi(\eta)\big) - A_1\mu\,,$$

$$v''(\eta) = 2A_1\, exp\big(-3\phi(\eta)\big) + 3A_1\lambda\, exp\big(-2\phi(\eta)\big) + A_1\lambda^2 exp\big(-\phi(\eta)\big) + 2A_1\mu\, exp\big(-\phi(\eta)\big) + A_1\lambda\mu,$$

$$v^3(\eta) = A_0^3 + 3A_0^2A_1 exp\big(-\phi(\eta)\big) + 3A_0A_1^2 exp\big(-2\phi(\eta)\big) + A_1^3\, exp\big(-3\phi(\eta)\big),$$

Inserting v, v'', v^3 into Eq. (16) and then equating the coefficients of like power of these polynomials to zero, we obtain the following nonlinear system of algebraic equations:

$$\left.\begin{aligned} 2A_1 - \frac{1}{2}A_1^3 &= 0 \\ 3A_1\lambda + \frac{3}{2}cA_1^2 - \frac{3}{2}A_0A_1^2 &= 0 \\ 2A_1\mu + A_1\lambda^2 + 3c\,A_0A_1 - \frac{3}{2}\,A_0^2A_1 - c^2A_1 &= 0 \\ A_1\mu\lambda - \frac{1}{2}A_0^3 + \frac{3}{2}cA_0^2 - c^2A_0 &= 0 \end{aligned}\right\}. \tag{18}$$

Solving system (18), we have the following results:

Case 1.

$$c = -\sqrt{\lambda^2 - 4\mu},\ \ A_0 = \pm\lambda - \sqrt{\lambda^2 - 4\mu}\ \text{ and } A_1 = \pm 2,$$

where λ and μ are arbitrary constants.

Case 2.

$$c = \sqrt{\lambda^2 - 4\mu},\ \ A_0 = \pm\lambda + \sqrt{\lambda^2 - 4\mu} \text{ and } A_1 = \pm 2,$$

where λ and μ are arbitrary constants.

Case I: Now substituting the values of $c,\ A_0,\ A_1$ into Eq. (17)

$$v(\eta) = \pm(\lambda + 2\,exp(-\phi(\eta))) - \sqrt{\lambda^2 - 4\mu}\,, \tag{19}$$

where $\eta = x + \sqrt{(\lambda^2 - 4\mu)}t,\ \lambda$ and μ are arbitrary constants.

Therefore, substituting Eqs. (6) to (11) into Eq. (19) respectively, we obtain the traveling wave solutions of the shallow water wave equation as follows:

when $\mu \neq 0,\ \lambda^2 - 4\mu > 0$, we obtain the solution,

$$v_{1,2}(\eta) = \pm\left(\lambda + \frac{4\mu}{\sqrt{(\lambda^2-4\mu)}\,tanh\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\eta+E)\right)+\lambda}\right) - \sqrt{\lambda^2 - 4\mu}\,, \tag{20}$$

where $\eta = x + \sqrt{(\lambda^2 - 4\mu)}t$ and E is an arbitrary constant.

when $\mu = 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu > 0$, we obtain the solution,

$$v_{3,4}(\eta) = \pm\,\lambda\left(1 + \frac{2}{exp(\lambda(\eta+E))-1}\right) - \sqrt{\lambda^2 - 4\mu}, \tag{21}$$

where $\eta = x + \sqrt{(\lambda^2 - 4\mu)}t$ and E is an arbitrary constant.

when $\mu \neq 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu = 0$, we obtain the solution,

$$v_{5,6}(\eta) = \left(\lambda - \frac{2\lambda^2(\eta+E)}{(\lambda(\eta+E))+2}\right) - \sqrt{\lambda^2 - 4\mu}\,, \tag{22}$$

where $\eta = x + \sqrt{(\lambda^2 - 4\mu)}t$ and E is an arbitrary constant.

when $\mu = 0,\ \lambda = 0,\ \lambda^2 - 4\mu = 0$, we obtain the solution

$$v_{7,8}(\eta) = \pm(\lambda + \frac{2}{(\eta+E)}), \tag{23}$$

where $\eta = x + \sqrt{(\lambda^2 - 4\mu)}t$ and E is an arbitrary constant.

Case II: Now substituting the values of $c,\ A_0,\ A_1$ into Eq. (17)

$$v(\eta) = \pm(\lambda + 2\,exp(-\phi(\eta))) + \sqrt{\lambda^2 - 4\mu}, \tag{24}$$

where $\eta = x - \sqrt{(\lambda^2 - 4\mu)}t$, and $\lambda,\ \mu$ and E are arbitrary constants.

Therefore, substituting Eqs. (6) to (11) into Eq. (24) respectively, we obtain three types of following traveling wave solutions of shallow water wave equation as follows:

when $\mu \neq 0,\ \lambda^2 - 4\mu > 0$, we obtain the solution,

$$v_{9,10}(\eta) = \pm\left(\lambda + \frac{4\mu}{\sqrt{(\lambda^2-4\mu)}\,tanh\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\eta+E)\right)+\lambda}\right) + \sqrt{\lambda^2-4\mu}, \tag{25}$$

where $\eta = x - \sqrt{(\lambda^2-4\mu)}t$ and E is an arbitrary constant.

when $\mu = 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu > 0$, we obtain the solution,

$$v_{11,12}(\eta) = \pm\lambda\left(1 + \frac{2}{exp(\lambda(\eta+E))-1}\right) + \sqrt{\lambda^2-4\mu}, \tag{26}$$

where $\eta = x - \sqrt{(\lambda^2-4\mu)}t$ and E is an arbitrary constant.

when $\mu \neq 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu = 0$, we obtain the solution,

$$v_{13,14}(\eta) = \pm\left(\lambda - \frac{2\lambda^2(\eta+E)}{(\lambda(\eta+E))+2}\right) + \sqrt{\lambda^2-4\mu}, \tag{27}$$

where $\eta = x - \sqrt{(\lambda^2-4\mu)}t$ and E is an arbitrary constant.

when $\mu = 0,\ \lambda = 0,\ \lambda^2 - 4\mu = 0$, we obtain the solution

$$v_{15,16}(\eta) = \pm(\lambda + \frac{2}{(\eta+E)}), \tag{28}$$

where $\eta = x - \sqrt{(\lambda^2-4\mu)}t$ and E is an arbitrary constant.

3.2 The generalized Klein-Gordon equation

In this section, we apply the of $\exp\left(-\phi(\eta)\right)$-expansion method to construct the exact traveling wave solutions of the generalized Klein-Gordon equation.

Consider the generalized Klein-Gordon equation [38-41],

$$v_{tt} + \alpha\, v_{xx} + \beta v + \gamma v^3 = 0, \tag{29}$$

here, $v(x,t)$ represents the particle wave profile at any varied instances, and $\alpha,\ \beta\ and\ \gamma$ are nonzero real arbitrary constants. Eq. (30) is also known reaction-diffusion equation and describe the solitary wave equation [42].

To look for new traveling wave solution of Eq. (11), we use $v(x,t) = v(\eta),\ \eta = x - ct$. Then Eq. (11) is reduced to the following nonlinear ordinary differential equation:

$$(c^2 + \alpha)v'' + \beta v + \gamma v^3 = 0. \tag{30}$$

Taking Eq. (5) in Eq. (30) and balancing the higher order derivative for the linear term v'' with the nonlinear term of the highest order v^3, we have $3n = n + 2$. Therefore, we get $n = 1$.

Therefore, the solution of Eq. (30) can be expressed by a polynomial in $exp\left(-\phi(\eta)\right)$ as follows:

$$v(\eta) = A_0 + A_1\left(exp\left(-\phi(\eta)\right)\right), \tag{31}$$

whereas $v(\eta)$ is a solution of Eq. (29) and A_0 and A_1 are constants to be determined later such that $A_N \neq 0$, while λ, μ are arbitrary constants. It is easy to see that

$v'(\eta) = -A_1\, exp\big(-2\phi(\eta)\big) - A_1\lambda\, exp\big(-\phi(\eta)\big) - A_1\mu.$

$v''(\eta) = \ 2A_1\, exp\big(-3\phi(\eta)\big) + 3A_1\lambda\, exp\big(-2\phi(\eta)\big) + A_1\lambda^2 ex\,p\big(-\phi(\eta)\big) + 2A_1\mu\, exp\big(-\phi(\eta)\big) + A_1\lambda\mu,$

$v^3(\eta) = A_0^3 + 3A_0^2 A_1 ex\,p\big(-\phi(\eta)\big) + 3A_0 A_1^2 ex\,p\big(-2\phi(\eta)\big) + A_1^3\, exp\big(-3\phi(\eta)\big).$

Inserting v, v'', v^3 into Eq. (30) and then equating the coefficients of like power of these polynomials to zero, we obtain the following a set of algebraic equations:

$$\left.\begin{array}{r} 2A_1\,\alpha + 2A_1V^2 + \gamma A_1^3 = 0 \\ 3A_1V^2\lambda + 3A_1\lambda\alpha + 3\gamma A_0A_1^2 = 0 \\ A_1\alpha\lambda^2 + 3\beta_2A_0^2A_1 + 2A_1V^2\mu + A_1V^2\lambda + A_1V^2\lambda^2 + A_1\beta + 2A_1\alpha\mu = 0 \\ \gamma A_0^3 + A_1V^2\mu\lambda + A_1\mu\lambda\alpha + \beta A_0 = 0 \end{array}\right\}. \tag{32}$$

Solving the above equations, we obtain

$$c = \pm\sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta}{4\mu-\lambda^2}},\ \ A_0 = \pm\lambda\sqrt{\frac{\beta}{(4\mu-\lambda^2)\,\gamma}}\ \ \text{and} \qquad A_1 = \pm 2\sqrt{\frac{\beta}{(4\mu-\lambda^2)\,\gamma}},$$

where $\alpha, \lambda, \mu, \beta$ and γ are arbitrary constants.

Now substituting the values of $V,\ A_0,\ A_1$ into Eq. (31) yields

$$v(\eta) = \pm\sqrt{\frac{\beta}{(4\mu-\lambda^2)\gamma}}\,\big(\lambda + 2\times exp\big(-\phi(\eta)\big)\big), \tag{33}$$

where $\eta = x \pm \sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta}{4\mu-\lambda^2}}\,t$, and $\alpha,\ \lambda,\ \mu,\ \beta$ and γ are arbitrary constants.

Therefore, substituting Eqs. (6) to (11) into Eq. (33) respectively, we get the following traveling wave solutions of the generalized Klein-Gordon equation as follows:

when $\mu \neq 0,\ \lambda^2 - 4\mu > 0$, we obtain the solution

$$v_{17,18}(\eta) = \pm\sqrt{\frac{\beta}{(4\mu-\lambda^2)\gamma}}\left(\lambda - \frac{4\mu}{\sqrt{(\lambda^2-4\mu)}\, tanh\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\eta+E)\right)+\lambda}\right) \tag{34}$$

where $\eta = x \pm \sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta}{4\mu-\lambda^2}}\,t$, and $\alpha,\ \lambda,\ \mu,\ \beta$ and γ are arbitrary constants.

when $\mu \neq 0,\ \lambda^2 - 4\mu < 0$, we obtain the trigonometric solutions

$$v_{19,20}(\eta) = \pm\sqrt{\frac{\beta}{(4\mu-\lambda^2)\,\gamma}}\left(\lambda + \frac{4\mu}{\sqrt{(4\mu-\lambda^2)}\, tan\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\eta+E)\right)-\lambda}\right) \tag{35}$$

where $\eta = x \pm \sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta}{4\mu-\lambda^2}}\,t$, and $\alpha,\ \lambda,\ \mu,\ \beta$ and γ are arbitrary constants.

When $\mu = 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu > 0,$

$$v_{21,22}(\eta) = \pm\sqrt{\frac{-\beta}{\gamma}}\left(1 + \frac{2}{exp(\lambda(\eta+E))-1}\right), \tag{36}$$

where $\eta = x \pm \sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta}{4\mu-\lambda^2}}t$, and $\alpha,\ \lambda,\ \mu,\ \beta$ and γ are arbitrary constants.

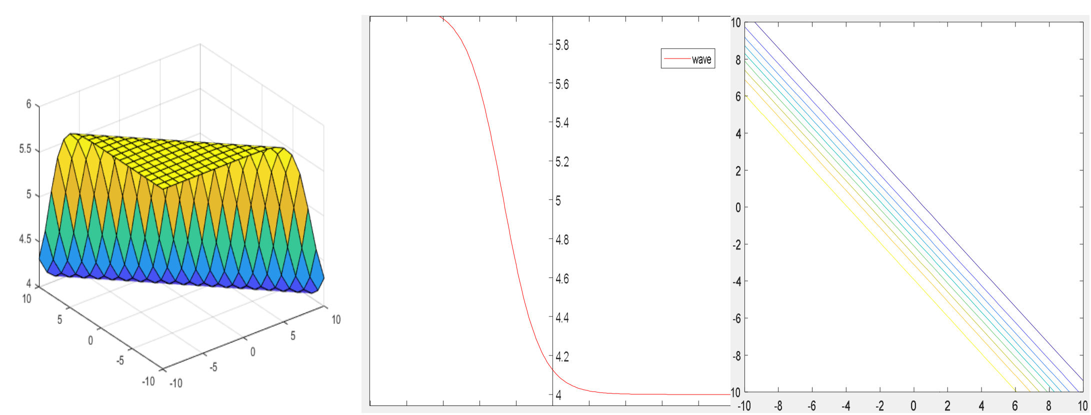
4 Graphical Representation and Physical Explanations

In this section, we will discuss the physical explanation and graphical representation of the obtained solutions by nonlinear shallow water wave equation and the generalized Klein-Gordon equation via an analytical technique, the $exp\ ) - \phi(\eta))$ -expansion method. The findings ae summarized and discussed in the following subsequent section.

4.1 The shallow water wave equation

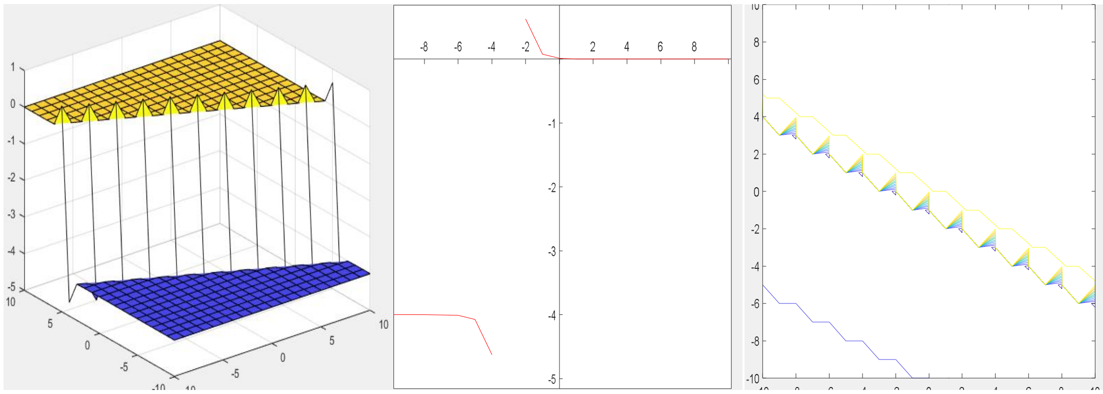
In this sub-section, we examine the nature of some obtained solutions of the shallow water wave equation (12) by selecting particular values of the parameters to visualize the exact solution to the physical phenomena. The obtained solutions of the shallow water wave equation incorporate of explicit solitary wave solutions namely hyperbolic function, exponential function and rational function solutions. We have depicted some graphical representation including $2D$, $3D$, and contour plot graph of the kink soliton solutions and singular kink soliton solutions by substituting the specific values of the unknown constants. From these explicit results we observe that solutions $v_1(\eta)$, $v_3(\eta)$, $v_5(\eta)$ and $v_7(\eta)$ are soliton solutions are shown in Figs. 1-4, respectively. For some special values of the physical parameters, the traveling wave solutions originated from the obtained exact explicit solutions as follows:

Solution (20) and (25) corresponding to the fixed values $\lambda = 3, \mu = 2, E = 1$ and $t = 1,$ within the interval $-10 \leq x, t \leq 10$ represented the exact solitary wave solution of kink type which shown in Fig. 1. Solution (21) and (26) corresponding to the fixed values $\lambda = 2, \mu = 0, E = 1$ and $t = 1,$ within the interval $-10 \leq x, t \leq 10$ represented the exact solitary wave solution of kink type which shown graphically in Fig. 2. Solution (22) and (27) corresponding to the fixed values $\lambda = 2, \mu = 1, E = 1$ and $t = 1,$ within the interval $-10 \leq x, t \leq 10$ represented the exact solitary wave solution of single soliton type which shown graphically in Fig. 3. Solution (23) and (28) corresponding to the fixed values $\lambda = 0, \mu = 0, E = 1$ and $t = 1,$ within the interval $-10 \leq x, t \leq 10$ represented the exact solitary wave solution of single soliton type which shown graphically in Fig. 4.



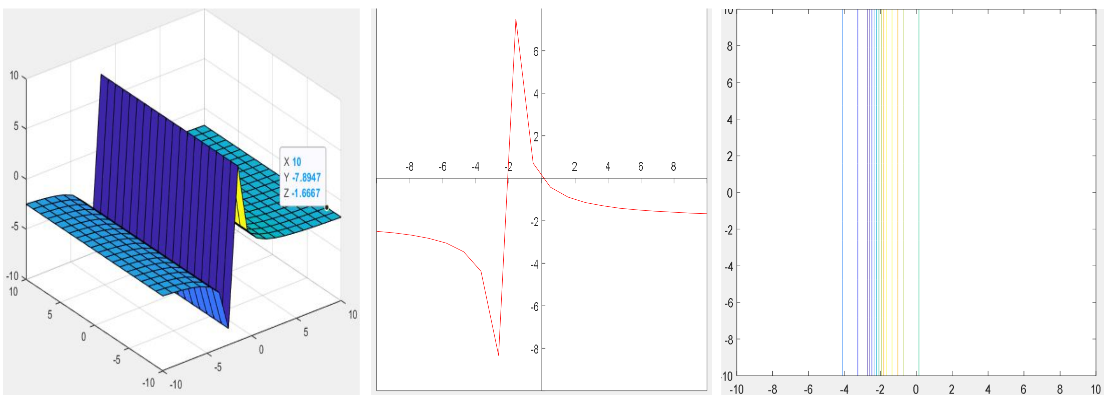
a. 3D graph b. 2D graph c. Contour Plot

Fig. 1. Graphical representation of the solution in $v_1(x, t)$ and projection at $t = 1$ for the unknown parameters $\lambda = 3,\ \mu = 1,\ E = 1$ within the interval $-10 \leq x, t \leq 10$



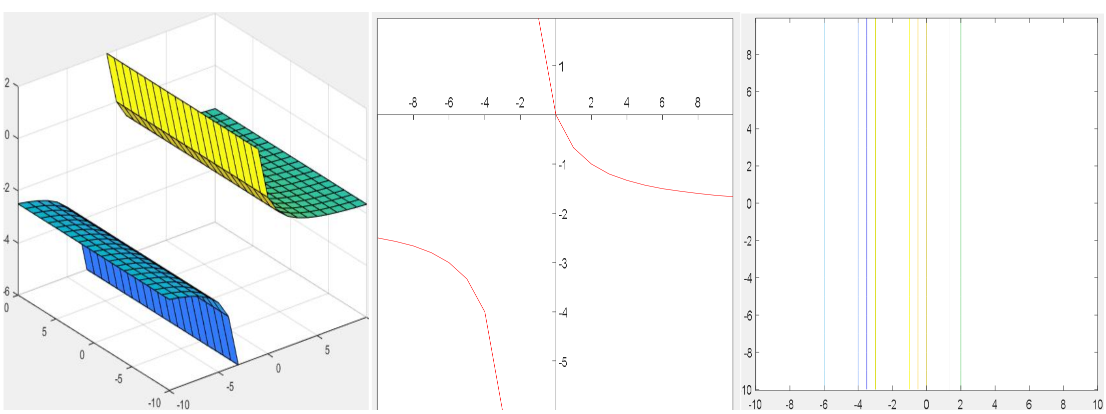
a. 3D graph b. 2D graph c. Contour Plot

Fig. 2. Graphical representation of the solution in $v_3(x,t)$ and projection at t=1 for the unknown parameters $\lambda = 2,\ \mu = 0,\ \ E = 1$ within the inter $-10 \le x, t \le 10$



a. 3D graph b. 2D graph c. Contour Plot

Fig. 3. Graphical representation of the solution in $v_5(x,t)$ and projection at t=1 for the unknown parameters $\lambda = 2,\ \mu = 1,\ \ E = 1$ within the inter $-10 \le x, t \le 10$

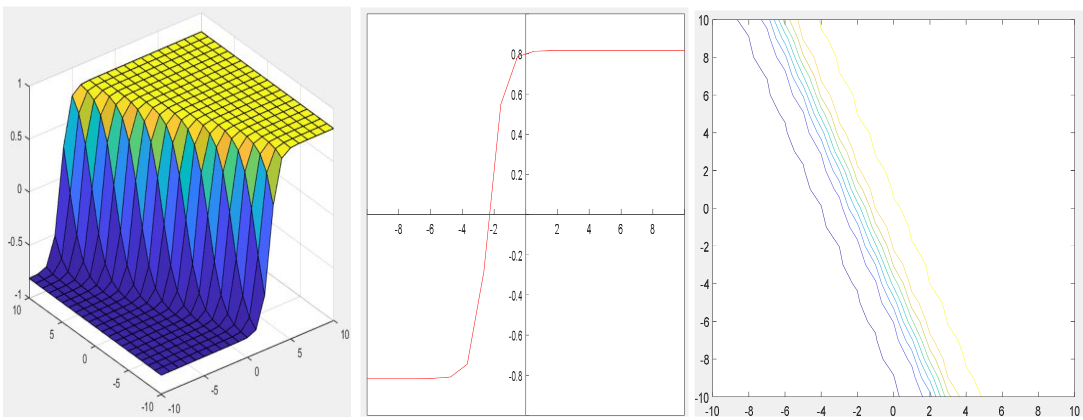


a. 3D graph b. 2D graph c. Contour Plot

Fig. 4. Graphical representation of the solution in $v_7(x,t)$ and projection at t=1 for the unknown parameters $\lambda = 0,\ \mu = 0,\ \ E = 1$ within the inter $-10 \le x, t \le 10$

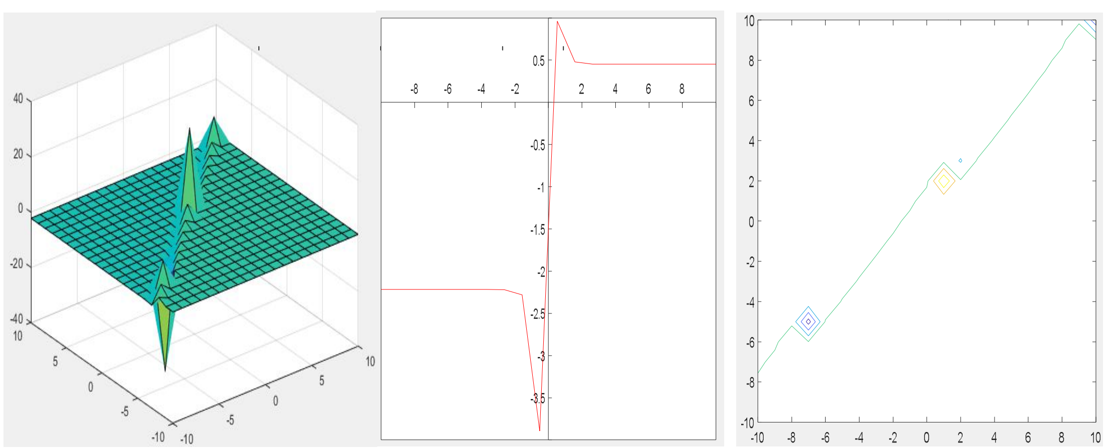
4.2 The generalized Klein-Gordon equation

In this section, the obtained solutions of the generalized Klein-Gordon equation incorporate four types of explicit solutions namely hyperbolic function, trigonometric function, rational function and exponential function solutions. It demonstrates the coupling between dissipation effect of the terms v_{tt}, v_{xx} and only one of the convection processes of γv^3. Eq. (30) combines only one of the nonlinear γu^3 and dissipation effect of the terms u_{tt}, u_{xx}. We have depicted some graphical representation including $2D$, $3D$, and contour plot graph of the kink soliton solutions and singular kink soliton solutions by substituting the specific values of the unknown constants. From these explicit results we observe that solutions $v_{17}(\eta)$ is kink solution and $v_{20}(\eta)$, $v_{21}(\eta)$ are periodic soliton solutions by setting suitable values of physical parameters which are shown in Figs 5-7, respectively. The solitary wave moves towards right if the velocity is positive and towards the left if the velocity is negative. The amplitudes and velocities are controlled by various physical parameters. It is concluded that the solitary waves for various values of physical and additional free parameters are highlighted by the graphical outcomes.



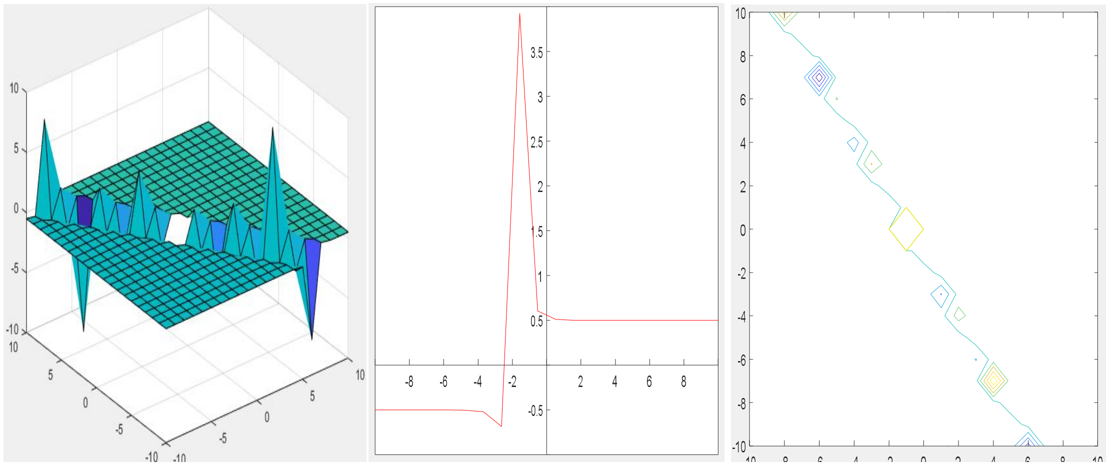
a. 3D graph b. 2D graph c. Contour Plot

Fig. 5. Graphical representation of the solution in $v_{17}(x,t)$ and projection at $t = 1$ for the unknown parameters $\beta = -2$, $\gamma = 3$, $\lambda = 3$, $\mu = 1$, $\alpha = -1$, $E = 1$ within the interval $-10 \le x, t \le 10$



a. 3D graph b. 2D graph c. Contour Plot

Fig. 6. Graphical representation of the solution in $v_{20}(x,t)$ and projection at $t = 1$ for the unknown parameters $\beta = -1$, $\gamma = 4$, $\lambda = 1$, $\mu = 2$, $E = 1$ within the interval $-10 \le x, t \le 10$



a. 3D graph b. 2D graph c. Contour Plot

Fig. 7. Graphical representation of the solution in $v_{21}(x,t)$ and projection at t=1 for the unknown parameters $\beta = -1,\ \gamma = -1,\ \lambda = 2,\ \mu = 0,\ \ E = 1$ within the inter $-10 \le x, t \le 1$

Fig. 5 shows the kink solitary waves of solution (34) by taking various physical parameters, $\lambda = 3$, $\mu = 2, E = 1$ and $t = 1$, within the interval $-10 \le x, t \le 10$.

Fig. 6 shows periodic solitary waves of solution (35) by taking various physical parameters $\lambda = 2$, $\mu = 0, E = 1$ and $t = 1$ within the interval $-10 \le x, t \le 10$.

Fig. 7 shows periodic solitary waves of solution (36) by taking various physical parameters, $\lambda = 2$, $\mu = 1, E = 1$ and $t = 1$ within the interval $-10 \le x, t \le 10$.

The solitary wave solution might be useful in analyzing the propagation of long waves in shallow water, iron sound waves in plasma, and vibrations in a nonlinear string.

5 Conclusion

In this paper, the exp $(-\phi(\eta))$-expansion method has been successfully implemented to solve the system of shall water wave equations and the generalized Klein-Gordon equation which are two of the most fascinating problems of modern mathematical physics. The method is quite efficient, straightforward, concise and practically well suited for use in finding nonlinear partial differential equations. It is noted that we found traveling wave solution in terms of hyperbolic, trigonometric, exponential and rational functions. The solutions can be useful in many circumstances, such as analyze the propagation of gravity waves in ocean, liquid flow, fluid flow in elastic tubes, waves in rivers and lakes in a smaller domain, etc. Due to the good performance of the exp $(-\phi(\eta))$-expansion method, it can be concluded that this method is reliable and proposes a variety of exact solutions of nonlinear evolution equations in the field in theoretical physics, mathematical physics and other branches of nonlinear sciences.

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Competing Interests

Authors have declared that no competing interests exist.

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