Asian Journal of Probability and Statistics

7(1): 1-15, 2020; Article no.AJPAS.55572

ISSN: 2582-0230

On the Topp Leone Exponentiated-G Family of Distributions: Properties and Applications

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Authors' contributions

This work was carried out in collaboration among all authors. Author SI designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors SID, IA and JHM managed the analyses of the study and the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2020/v7i130170
Editor(s):
(1) Dr. Oguntunde, Pelumi Emmanuel, Covenant University, Nigeria.
Reviewers:
(1) Xingting Wang, Howard University, USA.
(2) Terna Godfrey Ieren, Modibbo Adama University of Technology, Yola, Nigeria.
Complete Peer review History: http://www.sdiarticle4.com/review-history/55572

Received: 11 January 2020
Accepted: 20 March 2020
Published: 28 March 2020

Original Research Article

Abstract

We proposed a new family of distributions called the Topp Leone exponentiated-G family of distributions with two extra positive shape parameters, which generalizes and also extends the Topp Leone-G family of distributions. We derived some mathematical properties of the proposed family including explicit expressions for the quantile function, ordinary and incomplete moments, generating function and reliability. Some sub-models in the new family were discussed. The method of maximum likelihood was used to estimate the parameters of the sub-model. Further, the potentiality of the family was illustrated by fitting two real data sets to the mentioned sub-models.

Keywords: Hazard rate; reliability; Kumarasway; Lehmann alternative; order statistics.

1 Introduction

Any statistical analysis depends greatly on the statistical model used to represent the phenomena under study. Hence, the larger the class of statistical models available to the statistician the easier it is to choose a

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model. A quick survey of the models in common use reveals the abundance of statistical models in the literature. The usefulness of statistical distributions in several areas of research includes: modeling environmental pollution in environmental science, modeling duration without claims in actuarial science, modeling machine life cycle in engineering, modeling survival times of patients after surgery in the medical science, modeling failure rate of software in computer science and average time from marriage to divorce in the social science. However, the data generating process in many of these areas are characterized with varied degrees of skewness and kurtosis. Also, the data may exhibit non-monotonic failure rates such as the bathtub, unimodal or modified unimodal failure rates. Hence, modeling the data with the existing classical distributions does not provide a reasonable parametric fit and is often an approximation rather than reality.

Literature of lifetime distributions is rich with various continuous univariate distributions and still growing rapidly. Several extensions of some well-known lifetime distributions have been developed during the last two decades for modeling and analysis of many types of real life data that are having different random nature. This development is followed by many approaches for generating new families of distributions because the family contains many distributions and hence increase the chance of modeling a large number of real data. The techniques for modifying the classical distributions are usually referred to as generators in literature and are capable of improving the goodness-of-fit of the modified distributions. Some well-known generators are Marshal-Olkin generated family (MO-G) by Marshal and Olkin [1], the Beta-G by Eugene et al. [2] and Jones [3], Generalized Beta-generated distributions by Alexander et al. [4], Gamma-G (type 1) by Zografos and Balakrishanan [5], Gamma-G (type 2) by Ristic and Balakrishanan [6], Log-gamma-G by Amini et al.[7], Exponentiated generalized-G by Cordeiro et al. [8], Transformed-Transformer (T-X) by Alzaatreh et al. [9], exponentiated (T-X) by Alzaghal et al. [10], Weibull-G by Bourguignon et al. [11], Exponentiated half logistic generated family by Cordeiro et al. [12], Lomax-G by Cordeiro et al. [13], Kumaraswamy Odd log-logistic-G by Alizadeh et al. [14], Kumaraswamy Marshall-Olkin by Alizadeh et al. [15], Beta Marshall-Olkin by Alizadeh et al. [16], Kummer-beta generalized distributions by Pescimet al. [17], A new family of Marshall–Olkin extended distributions by Alshangiti et al. [18], A new family of distributions: Libby-Novick beta by Cordeiro et al. [19], Type 1 Half-Logistic family of distributions by Cordeiro et al. [20], The generalized transmuted-G family by Nofal et al. [21], Generalized transmuted family by Alizadeh et al. [22], Another generalized transmuted family by Merovci et al. [23], Transmuted exponentiated generalized-G family by Yousof et al. [24], Transmuted geometric G family by Afify et al. [25], Beta transmuted-H family by Afify et al. [26], Kumaraswamy transmuted-G family by Afify et al. [27], Topp–Leone Family of Distributions by Al-Shomrani et al. [28], The transmuted transmuted-G family by Mansour et al. [29], The Exponentiated Kumaraswamy-G Class by Silva et al. [30], The extended Weibull-G family of distributions by Korkmaz [31], The Exponentiated Generalized Topp Leone-G Family of distributions by Reyad et al. [32].

The motivation for generalizing distributions for modeling lifetime data lies in the flexibility to model both monotonic and non-monotonic failure rates even though the baseline failure rate may be monotonic. The basic justifications for generating a new family of distributions in practice are the following: to produce a skewness for symmetrical models; to generate distributions with left-skewed, right-skewed, symmetric, or reversed-J shape; to define special models with all types of hazard rate function; to make the kurtosis more flexible compared to that of the baseline distribution; to construct heavy-tailed distributions for modeling various real data sets; to provide consistently better fits than other generated distributions with the same underlying model.

Exponentiated-G family:

$$G(x;\alpha) = [H(x)]^{\alpha} \tag{1}$$

Its corresponding pdf is given as:

$$g(x;\alpha) = \alpha h(x)[H(x)]^{\alpha-1}, \qquad x > 0 \tag{2}$$

$\alpha > 0$ is the shape parameter.

Where $H(x)$ and $h(x)$ are cdf and pdf of the baseline distribution.

Topp Leone –G family:

$$F(x) = \{1 - [1 - G(x)]^2\}^{\theta} \tag{3}$$

Its corresponding pdf is given as

$$f(x) = 2\theta g(x)[1 - G(x)][1 - \left(1 - G(x)\right)^2]^{\theta - 1},\ x > 0 \tag{4}$$

$\theta > 0$ is the shape parameter.

Where $G(x)$ and $g(x)$ are cdf and pdf of the baseline distribution.

In order to meet up with the basic justifications for generating new families of distributions, there is the need to combine (1) and (3) to form a single family so as to share the properties of both families and have two shape parameters. The added parameter will make the family of distribution to be more flexible to model both monotonic and non-monotonic failure rates.

The rest of the paper is outlined as follows. In Section 2, we define the TLEx-G family of distributions. In Section 3, we derive a very useful linear representation for the TLx-G density function. We obtain in Section 4 some general statistical properties of the proposed family including ordinary and incomplete moments, mean deviations, residual life function and reversed residual life function. We obtained the explicit expression for the quantile function in section 5. Order statistics are investigated in Section 6. In Section 7, maximum likelihood estimation (MLE) of the model parameters is investigated. In Section 8, two special models of this family are presented and some plots of their pdf's are given. In section 9, an application to a real dataset to illustrate the potentiality of the new family was presented. Finally, some concluding remarks are presented in Section 10.

2 The New Family

In this paper, we define a new family of distributions that extends the TL-G family called Topp Leone Exponentiated-G family of distributions and derive some of its structural properties.

Let the exponentiated-G family be the baseline family with cdf and pdf given in (1) and (2) respectively.

Then, the Topp Leone Exponentiated-G family has the cdf given as:

$$F(x;\ \alpha, \varphi) = \int_0^{[H(x)]^{\alpha}} 2\theta g(t)(1 - G(t))(1 - (1 - G(t))^2)^{\theta - 1}\, dt$$

Let $y = [H(t)]^{\alpha}$, $dy = \alpha h(t)[H(t)]^{\alpha - 1} dt = g(t; \alpha, \varphi)dt,$ when $t = 0, y = 0$; when $t = x, y = [H(x)]^{\alpha}$

So,

$$F(x;\ \alpha, \varphi) = 2\theta \int_0^{[H(x)]^{\alpha}} (1 - y)(1 - (1 - y)^2)^{\theta - 1}\, dy$$

$$F(x;\ \alpha, \varphi) = 2\theta \int_0^{[H(x)]^{\alpha}} (1 - y)(2y - y^2)^{\theta - 1}\, dy$$

Let $m = 2y - y^2$, $dm = 2(1 - y)dy$, when $y = 0, m = 0$; when $y = [H(x)]^{\alpha}$, m=$2[H(x)]^{\alpha} - [H(x)]^{2\alpha}$.

$$F(x;\alpha,\varphi)=\theta\int_0^{2[H(x)]^{\alpha}-[H(x)]^{2\alpha}}(m)^{\theta-1}\,dm$$

$$F(x;\alpha,\varphi)=\theta\left(\frac{m^{\theta}}{\theta}\right)_0^{2[H(x)]^{\alpha}-[H(x)]^{2\alpha}}$$

$$F(x;\alpha,\varphi)=\left(m^{\theta}\right)_0^{2[H(x)]^{\alpha}-[H(x)]^{2\alpha}}$$

$$F(x;\alpha,\varphi)=2[H(x)]^{\alpha}-[H(x)]^{2\alpha}-0$$

$$F(x;\alpha,\theta,\varphi)=\{1-[1-H(x,\varphi)^{\alpha}]^{2}\}^{\theta}, \tag{5}$$

where $\alpha>0$ and $\theta>0$ are two additional shape parameters to the G-family of distribution.

Its corresponding pdf is given as

$$\frac{dF(x;\ \alpha,\theta,\varphi)}{dx}=2\alpha\theta h(x;\varphi)H(x;\varphi)^{\alpha-1}[1-H(x;\varphi)^{\alpha}]\{1-[1-H(x;\varphi)^{\alpha}]^{2}\}^{\theta-1}$$

$$f(x;\alpha,\theta,\varphi)=2\alpha\theta h(x;\ \varphi)H(x;\ \varphi)^{\alpha-1}[1-H(x;\ \varphi)^{\alpha}]\{1-[1-H(x;\ \varphi)^{\alpha}]^{2}\}^{\theta-1} \tag{6}$$
$$x>0,\qquad \alpha,\qquad \theta,\qquad \varphi>0$$

3 Linear Representation

Now, we provide a useful representation for (5). Here, the infinite mixture representations for the cdf and pdf of the TLEx-G family are given in terms of baseline densities. Consider following series expansion

$$(1-y)^{b}=\textstyle\sum_{i=0}^{\infty}\binom{b}{i}(-1)^{i}y^{i} \tag{7}$$

Using the series expansion in (7), then (5) becomes

$$\{1-[1-H(x,\varphi)^{\alpha}]^{2}\}^{\theta}=\sum_{i=0}^{\infty}\binom{\theta}{i}(-1)^{i}[1-\mathrm{G(x)}^{\alpha}]^{2i}$$

Consider

$$[1-\mathrm{G(x)}^{\alpha}]^{2i}=\sum_{j=o}^{\infty}\binom{2i}{j}(-1)^{j}G(x)^{\alpha j}$$

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\theta}{i}\binom{2i}{j}(-1)^{i+j}G(x)^{\alpha j}$$

According to Jamal et al. [33], $G(x)^{\alpha j}=\sum_{q=0}^{\infty}\sum_{q=k}^{\infty}\binom{\alpha j}{k}\binom{k}{q}(-1)^{k+q}G(x)^{q}$

Now,

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\theta}{i}\binom{2i}{j}(-1)^{i+j}\sum_{q=0}^{\infty}\sum_{q=k}^{\infty}\binom{\alpha j}{k}\binom{k}{q}(-1)^{k+q}G(x)^{q}$$

Rewriting the above expression, we obtain an expansion for the cdf of the TLEx-G family

$$F(x) = \sum_{q=0}^{\infty} a_q H_q(x) \quad (8)$$

Where $a_q = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\theta}{i}\binom{2i}{j}(-1)^{i+j}\sum_{q=k}^{\infty}\binom{\alpha j}{k}\binom{k}{q}(-1)^{k+q},\ H_q(x) = G(x)^q$

Similarly, we have an expansion for the pdf of TLEx-G family as

$$f(x) = \sum_{q=0}^{\infty} a_q h_{q-1}(x), \quad (9)$$

Where, $h_{q-1}(x) = q.\,g(x)G(x)^{q-1}$.

Where $H_q(x)$ denotes the Exp-G cdf with power parameter q. Equation (8) reveals that the TLEx-G density function is a linear combination of Exp-G densities. Thus, some structural properties of the ExTL-G class such as the ordinary and incomplete moments and generating function can be obtained from well known Exp-G properties.

4 Mathematical Properties

In this section, we investigate some mathematical properties of the TLEx-G family of distributions. Established algebraic expansions to determine some structural properties of the ExTL-G family of distributions can be more efficient than computing those directly by numerical integration of its density function.

4.1 Moments

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). The r^{th} moment of TLEx-G family is given by

$$\mu_r' = E(x^r) = \int_0^{\infty} x^r f(x)dx \quad (10)$$

Using the infinite mixture representation of the pdf in equation (9), we have

$$\mu_r' = \sum_{q=0}^{\infty} a_q \emptyset_r', \quad (11)$$

Where $\emptyset_r' = \int_0^{\infty} x^r\, h_{q-1}(x)dx$

4.2 Generating function

The moment generating function (mgf) of X is given as

$$M_x(t) = \int_0^{\infty} e^{tx} f(x)dx \quad (12)$$

From the infinite mixture representation of pdf in equation (9), we obtain

$$M_x(t) = \sum_{q=0}^{\infty} a_q M_{q-1}(t) \quad (13)$$

Where $M_{q-1}(t) = \int_0^{\infty} x^r\, h_{q-1}(x)dx$

4.3 Incomplete moment

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The answers too many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well. The s^{th} incomplete moments, say $\varphi_s(t)$ is given by

$$\varphi_s(x) = \mu_s' = \int_0^x x^s f(x)dx \tag{14}$$

From the infinite mixture representation of pdf in equation (3.3), we get

$$\mu_s' = \sum_{q=0}^{\infty} a_q \tau_s' \tag{15}$$

Where $\tau_s' = \int_0^x x^s\, h_{q-1}(x)dx$

Note that the integrals $\emptyset_r'$, $M_{q-1}(t)$ and τ_s' depend only on any choice of baseline distribution.

The first incomplete moment of the TLEx-G family $\varphi_s(t)$ can be obtained by setting $s = 1$.

4.4 Mean deviation

The mean deviation about the mean $[\delta_2 = E(|x - \mu_1'|)]$ and about the median $[\delta_1 = E(|x - M|)]$ of the TLEx-G family are given as

$$\delta_1 = 2\mu_1' F(\mu_1') - 2J(\mu_1') \tag{16}$$

$$\delta_2 = \mu_1' - 2J(M) \tag{17}$$

where $\mu_1' = E(x)$, is the mean, $M = Median(x) = Q(0.5)$, is the median and $j(c) = \int_0^x xf(x)dx$. From the infinite mixture representation of pdf in equation (9), we get

$$j(c) = \sum_{q=0}^{\infty} a_q\, T_{q-1}'(c) \tag{18}$$

where $T_{q-1}'(c) = \int_0^c x\, h_{q-1}(x)dx$ the integral depends on G(x) and g(x).

4.5 Reliability function

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$R(x;\alpha,\theta) = 1 - F(x;\alpha,\theta)$$

$$R(x;\alpha,\theta) = 1 - \{1 - [1 - H(x)^{\alpha}]^2\}^{\theta} \tag{19}$$

4.6 Hazard function

The hazard function is given as

$$\tau(x;\alpha,\theta) = \frac{f(x;\alpha,\theta)}{R(x,\alpha,\theta)} = \frac{2\alpha\theta h(x)H(x)^{\alpha-1}[1-H(x)^{\alpha}]\{1-[1-H(x)^{\alpha}]^2\}^{\theta-1}}{1-\{1-[1-H(x)^{\alpha}]^2\}^{\theta}} \tag{20}$$

5 Quantile Function

The TLEx-G family is easily simulated by inverting (5) as follows: if u has a uniform U(0,1) distribution, then the solution of the nonlinear equation is given by

$$Q(u) = G^{-1}([1-(1-U^{\frac{1}{\theta}})^{\frac{1}{2}}]^{\frac{1}{\alpha}}) \tag{21}$$

That is, $x \sim TLEx - G(\varphi)$.

In particular, the median of the TLEx-G family of distributions can be derived by substituting u = 0.5 in Equation (23) as follows:

$$Q(0.5) = G^{-1}([1-(1-0.5^{\frac{1}{\theta}})^{\frac{1}{2}}]^{\frac{1}{\alpha}}) \tag{22}$$

6 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be n independent random variable from the ExTL-G family of distributions and let $x_{(1)} \le x_{(2)} \le, \ldots, \le x_{(n)}$ be their corresponding order statistics.

Let $F_{r:n}(x)$ and $f_{r:n}(x),\ r = 1,2,3,\ldots,n$ denote the cdf and pdf of the rth order statistics $X_{r:n}$ respectively. The pdf of $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}$$

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} 2\alpha\theta h(x)H(x)^{\alpha-1}$$

$$\text{X } [1-H(x)^{\alpha}]\{1-[1-H(x)^{\alpha}]^2\}^{\theta-1}[\{1-[1-H(x)^{\alpha}]^2\}^{\theta}]^{r-1}\left[1-\{1-[1-H(x)^{\alpha}]^2\}^{\theta}\right]^{n-r} \tag{23}$$

Equation (24) is the r^{th} order statistic from the ExTL-G family of distributions.

The pdf of the maximum order statistics is obtained by setting $r = n$ as

$$f_{n:n}(x) = n2\alpha\theta h(x)H(x)^{\alpha-1}[1-H(x)^{\alpha}]\{1-[1-H(x)^{\alpha}]^2\}^{\theta-1}[\{1-[1-H(x)^{\alpha}]^2\}^{\theta}]^{n-1}$$

Also the pdf of the minimum order statistics is obtained by setting $r = 1$ as

$$f_{1:n}(x) = n2\alpha\theta h(x)H(x)^{\alpha-1}[1-H(x)^{\alpha}]\{1-[1-H(x)^{\alpha}]^2\}^{\theta-1}\left[1-\{1-[1-H(x)^{\alpha}]^2\}^{\theta}\right]^{n-1}$$

7 Estimation

Here, we consider the estimation of the unknown parameters of the TLEx-G family by the maximum likelihood method for the complete samples. The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and deliver simple approximations that work well in finite samples. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically.

Let $x_{1,}x_2,\ldots,x_n$ be an iid observed random sample of size n from the TLEx-G family. Then, the log-likelihood function based on observed sample for the vector of parameter $\emptyset = (\alpha,\theta,\varphi)^T$ is given by

$$l(\emptyset) = nlog2 + nlog\alpha + nlog\theta + \sum_{i=1}^{n}\log[h(x_i;\varphi)] + (\alpha-1)\sum_{i=1}^{n}\log[H(x_i;\varphi)] +$$

$$\sum_{i=1}^{n} \log(1 - H(x_i;\varphi)^{\alpha}) + (\theta - 1)\sum_{i=1}^{n} \log\,(1 - (1 - H(x_i;\varphi)^{\alpha})^2)$$

The components of score vector $U = (U_\alpha, U_\theta, U_\varphi)^T$ are given as

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \log[H(x_i;\varphi)] + \sum_{i=1}^{n}\left[\frac{H(x_i;\varphi)^{\alpha}\log[H(x_i;\varphi)]}{1-H(x_i;\varphi)^{\alpha}}\right] + (\theta-1)2\sum_{i=1}^{n}\left[\frac{[1-H(x_i;\varphi)^{\alpha}]H(x_i;\varphi)^{\alpha}\log[H(x_i;\varphi)]}{1-[1-H(x_i;\varphi)^{\alpha}]^2}\right]$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} \log\,[1-(1-H(x_i;\varphi)^{\alpha})^2]$$

$$U_\varphi = \sum_{i=1}^{n}\left[\frac{h(x_i;\varphi)^{\varphi}}{h(x_i;\varphi)}\right] + (\alpha-1)\sum_{i=1}^{n}\left[\frac{H(x_i;\varphi)^{\varphi}}{H(x_i;\varphi)}\right] - \alpha\sum_{i=1}^{n}\left[\frac{[1-H(x_i;\varphi)^{\varphi}]H(x_i;\varphi)^{\alpha-1}H(x_i;\varphi)^{\varphi}}{1-H(x_i;\varphi)^{\alpha}}\right] - 2(\theta-1)\sum_{i=1}^{n}\left[\frac{H(x_i;\varphi)^{\alpha-1}H(x_i;\varphi)^{\varphi}}{1-[1-H(x_i;\varphi)^{\alpha}]^2}\right]$$

Setting U_α, U_θ and U_φ equal to zero and solving these equations simultaneously yields the MLEs. These equations cannot be solved analytically, and analytical software are required to solve them numerically.

8 Some Sub-models of the TLEX-G Family of Distributions

In this section, we provide examples of the TLEx-G family. The pdf of the TLEx-G family will be most tractable when *f(x)* and *F(x)* have simple analytic expressions. These special models generalize some well-known distributions reported in the literature. Here, we provide two special models of this family corresponding to the baseline Exponential (Ex) and Log-logistic (LL) distributions to show the flexibility of the new family.

8.1 The TLEx Exponential (TLExEx) distribution

The parent exponential distribution has cdf and pdf given as

$$H(x;\beta) = 1 - e^{-\beta x} \tag{24}$$

$$h(x;\beta) = \beta e^{-\beta x} \tag{25}$$

The cdf and pdf of TLExEx distribution are given by

$$F(x;\alpha,\beta) = \{1 - (1 - \left(1 - e^{-\beta x}\right)^{\alpha})^2\}^{\theta} \tag{26}$$

$$f(x;\alpha,\beta) = 2\alpha\theta\beta e^{-\beta x}[1 - e^{-\beta x}]^{\alpha-1}[1 - \left(1 - e^{-\beta x}\right)^{\alpha}]\{1 - (1 - \left(1 - e^{-\beta x}\right)^{\alpha})^2\}^{\theta-1},\ x \geq 0 \tag{27}$$

$\alpha, \theta, \beta > 0$ are the shape parameters.

8.1.1 Properties of TLExEx distribution

8.1.1.1 Reliability function

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$R(x;\alpha,\theta,\beta) = 1 - F(x;\alpha,\theta,\beta)$$

$$R(x;\alpha,\theta,\beta) = 1 - \{1 - (1 - \left(1 - e^{-\beta x}\right)^{\alpha})^2\}^{\theta} \tag{28}$$

8.1.1.2 Hazard function

The hazard function is given as

$$\tau(x;\alpha,\theta,\beta) = \frac{f(x;\alpha,\theta,\beta)}{R(x,\alpha,\theta,\beta)} = \frac{2\alpha\theta\beta e^{-\beta x}[1-e^{-\beta x}]^{\alpha-1}[1-\left(1-e^{-\beta x}\right)^{\alpha}]\{1-(1-\left(1-e^{-\beta x}\right)^{\alpha})^2\}^{\theta-1}}{1-\{1-(1-\left(1-e^{-\beta x}\right)^{\alpha})^2\}^{\theta}} \tag{29}$$

8.1.1.3 Quantile function

The TLExEx distribution is easily simulated by inverting (27) as follows: if u has a uniform $U(0,1)$ distribution, then the solution of the nonlinear equation is given by

$$x = \frac{1}{\beta}\left\{-log\left[1 - \left(1 - \left(1 - u^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}\right]\right\} \tag{30}$$

The median of the TLExEx distribution is obtained by setting $u = 0.5$ in (31) as

$$x = \frac{1}{\beta}\left\{-log\left[1 - \left(1 - \left(1 - 0.5^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}\right]\right\} \tag{31}$$

8.1.1.4 Order statistics

Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, \ldots, n$ denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. The pdf of $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^i[F(x)]^{r+i-1}f(x)$$

Using the pdf and cdf of TLExEx distribution, we have

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} 2\alpha\theta\beta \sum_{i=0}^{n-r}\sum_{j,k,l=0}^{\infty}(-1)^{i+j+k+l}\binom{\theta(r+i)-1}{j}\binom{2j+1}{k}\binom{\alpha(k+1)-1}{l}\left[e^{-\beta x}\right]^{l+1} \tag{32}$$

Equation (33) is the pdf of the r^{th} order statistics of the TLExEx distribution from which we can obtain the minimum order statistics by setting $r = 1$ and maximum order statistics by setting $r = n$.

8.2 The TLEx Log-Logistic (TLExLL) distribution

The parent log-logistic distribution has cdf and pdf given as

$$H(x;\beta) = \frac{x^{\beta}}{1+x^{\beta}} \tag{33}$$

$$h(x;\beta) = \frac{\beta x^{\beta-1}}{(1+x^{\beta})^2} \tag{34}$$

The cdf and pdf of TLExLL distribution are given by

$$F(x;\alpha,\beta)=\left[1-(1-(\tfrac{x^{\beta}}{1+x^{\beta}})^{\alpha})^{2}\right]^{\theta-1} \qquad (35)$$

$$f(x;\alpha,\beta)=2\alpha\theta\frac{\beta x^{\beta-1}}{(1+x^{\beta})^{2}}\left[\frac{x^{\beta}}{1+x^{\beta}}\right]^{\alpha-1}[1-(\tfrac{x^{\beta}}{1+x^{\beta}})^{\alpha}]\{1-(1-(\tfrac{x^{\beta}}{1+x^{\beta}})^{\alpha})^{2}\}^{\theta-1} \qquad (36)$$

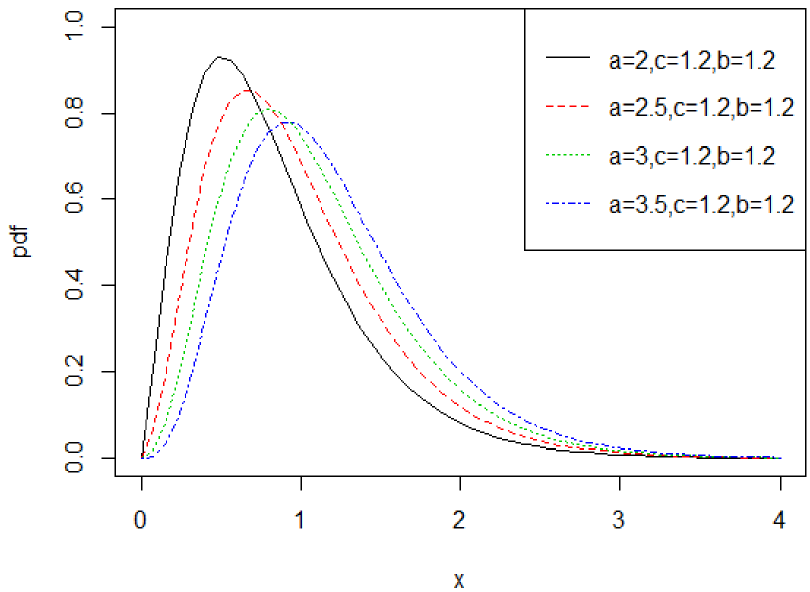

Fig. 1. Plot of the TLExEx pdf for some parameter values (a = α, b = β, c = θ)

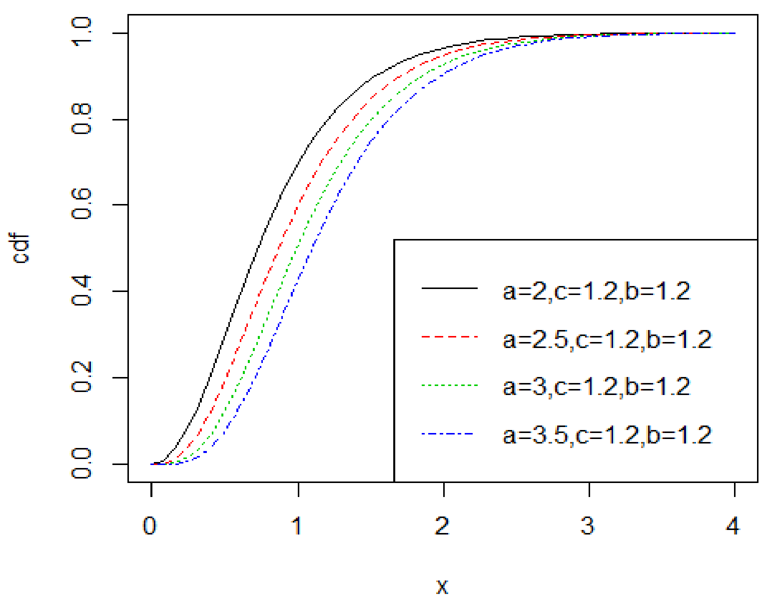

Fig. 2. Plot of the TLExEx cdf for some parameter values (a = α, b = β, c = θ)

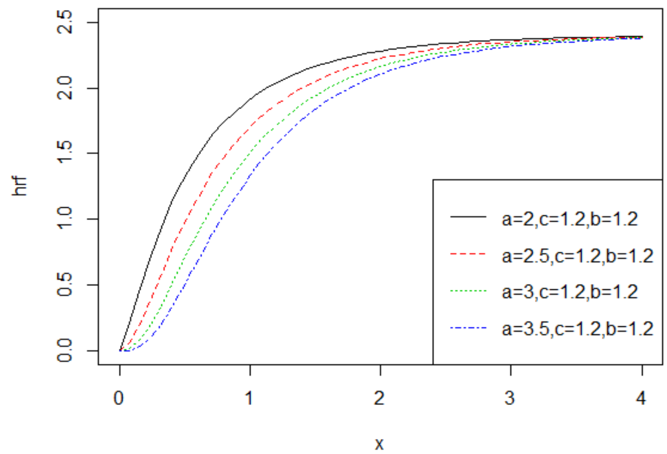

Fig. 3. Plot of the TLExEx hrf for some parameter values (a = α, b = β, c = θ)

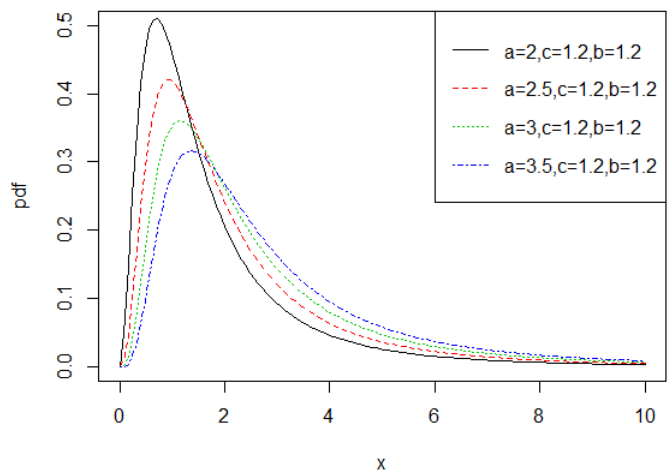

Fig. 4. Plot of the TLExLL pdf for some parameter values (a = α, b = β, c = θ)

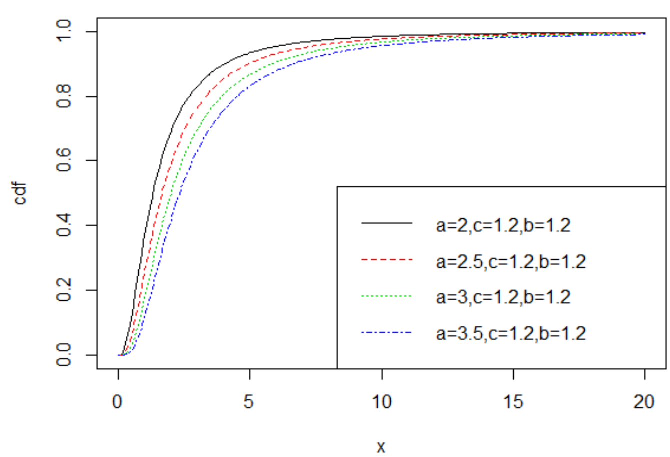

Fig. 5. Plot of the TLExLL cdf for some parameter values (a = α, b = β, c = θ)

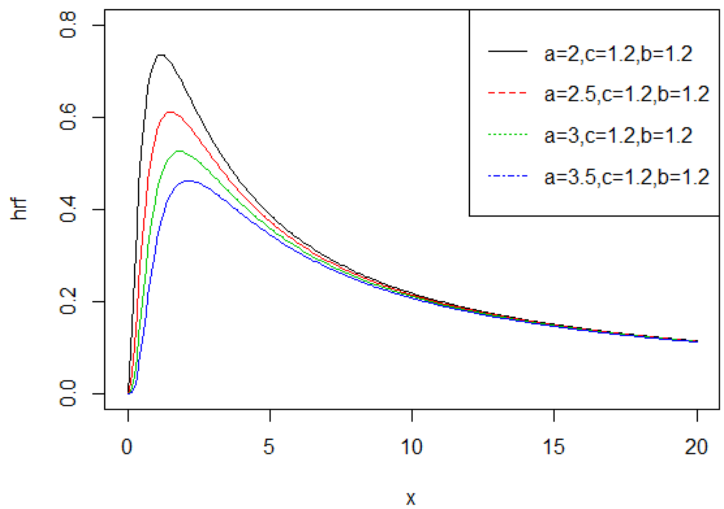

Fig. 6. Plot of the TLExLL hrt for some parameter values (a = α, b = β, c = θ)

9 Application

In this section, we fit the TLExEx distribution to two real data sets and for illustrative purposes also present a comparative study with the fits of TLEx, Ex, ExEx, Lx, and IEx models. These applications prove empirically the flexibility of the new family of distributions in modeling positive data. All the computations are performed using the R software.

Data set I:

The data set represents the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile. The data set has been previously used by Sickle-Santanello et al. [34]. The data are:

1,3,3,4,10,13,13,16,16,24,26,27,28,30,30,32,41,51,61,65,67,70,72,73,74,77,79,80,81,87,87,88,89,91,93,96,97,100,101,104,104,108,109,120,131,150,157,167,231,240,400.

Table 1. MLEs and their standard errors (in parentheses) for death times of patients' data

Distribution	Parameters	Log-lokelihood	AIC
TLExEx	$\hat{\alpha}$ = 0.7496 (0.0644)		
	$\hat{\theta}$ = 1.1768 (0.0249)	-212.3536	430.7072
	$\hat{\beta}$ = 0.0027 (0.0002)		
TLEx	$\hat{\theta}$ = 1.2220 (0.2293)		
	$\hat{\beta}$ = 0.0070 (0.0012)	-274.3936	552.787
Ex	$\hat{\beta}$ = 0.0124 (0.0017)	-274.9438	551.8875
ExEx	$\hat{\alpha}$ = 1.2220 (0.2289)	-274.3936	552.7871
	$\hat{\beta}$ = 0.0140 (0.0024)		
Lx	$\hat{\theta}$ = 11.130 (4.3580)	-275.0047	554.0093
	$\hat{\beta}$ = 0.0001 (0.0000)		
IEx	$\hat{\beta}$ = 17.3790 (2.4220)	-306.1066	614.2133

Data set II:

This data set represents the failure times of the air conditioning system of an airplane. The data set was given by Linhart and Zucchini [35] and it has also been used by Shanker et al., [36]. The data set is presented below:

23,261,87,7,120,14,62,47,225,71,246,21,42,20,5,12,120,11,3,14,71,11,14,11,16,90,1,16,52,95

Table 2. MLEs and their standard errors (in parentheses) for failure times of AC system data

Distribution	Parameters	Log-lokelihood	AIC
TLExEx	$\hat{\alpha}$ = 1.0551 (0.1237)		
	$\hat{\theta}$ = 1.1520 (0.0293)	-72.6538	151.3077
	$\hat{\beta}$ = 0.0018 (0.0002)		
TLEx	$\hat{\theta}$ = 1.1708 (0.0232)		
	$\hat{\beta}$ = 0.0074 (0.0000)	-85.2003	174.4007
ExEx	$\hat{\alpha}$ = 4.6140 (1.0735)	-95.2340	194.468
	$\hat{\beta}$ = 0.0132 (0.0028)		

10 Conclusion

A new family of continuous distributions called the Topp Leone Exponentiated-G (TLEx-G) class is introduced and studied. The proposed class contains two parameters more than those in the baseline distribution. Several new models can be generated based on this family by considering special cases for G. We demonstrated that the TLEx-G density function can be expressed as a linear combination of exponentiated-G (Exp-G) density functions. This result allows us to obtain general explicit expressions for some measures of the TLEx-G class such as the ordinary and incomplete moments, generating function and mean deviations. The method of maximum likelihood is applied to estimate the model parameters. Two real data sets are used to show that some models corresponding to the TLEx-G family of distributions gave a better fit compare to models corresponding to TL-G and Ex-G families of distributions.

Competing Interests

Authors have declared that no competing interests exist.

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