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Resultant-Based Method for Overdetermined Strata for Degree 4 hyperbolic Polynomials

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Abstract

We present a resultant-based method to calculate the overdetermined strata for degree 4 hyperbolic polynomials in one variable. It is a new method to calculate overdetermined strata. We present also the complete study of the overdetermined strata for degree 4 hyperbolic polynomials by the geometric method.

Keywords: Hyperbolic polynomials; Gegenbauer polynomials; overdetermined strata; resultants.
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1 Introduction

We consider the polynomial $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n,\ x, a_i \in \mathbb{R}$. This polynomial is called (*strictly*) *hyperbolic* if all its roots are real (real and distinct). If P is (strictly) hyperbolic, then such are $P^{(1)}$,..., $P^{(n-1)}$ as well. Examples of hyperbolic polynomials are the ones of all known orthogonal families (e.g. the Legendre, Laguerre, Hermite, Tchebychev polynomials).

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Some properties of hyperbolic polynomials and criteria of hyperbolicity have been studied at the beginning of the twentieth century, see [1]. The interest of hyperbolic polynomials appear in the theory of linear partial differential equations, see [2], and in the potential theory, see [3], [4], [5], [6] and [7].

If the coefficients of a polynomial depend on parameters, we say that the set of values taken by these parameters for which the polynomial is hyperbolic, is *the hyperbolicity domain*, denoted by Π^*. The change $x \mapsto x - a_1/n$ reduces the study of Π^* to the case $a_1 = 0$.

Lemma 1.1. *In the case $a_1 = 0$ the polynomial P is hyperbolic only if $a_2 \leq 0$. If $a_1 = a_2 = 0$, then P is hyperbolic only for $a_2 = \cdots = a_n = 0$.*

Proof. All derivatives of P must be hyperbolic, in particular $P^{(n-2)}(x) = (n!/2)x^2 + (n-2)!a_2$, therefore $a_2 \leq 0$.

Let $a_1 = a_2 = 0$. As $P^{(n-3)} = (n!/6)x^3 + (n-3)!a_3$ must be hyperbolic, one has $a_3 = 0$ etc. □

A second change $x = \sqrt{|a_2|}x$ can reduce the study of Π^* to the case $a_1 = 0$ and $a_2 = -1$. Denote by $\Pi = \Pi^* \cap \{a_1 = 0,\ a_2 = -1\}$. Hence, we consider from now on the family of polynomials of the form

$$P(x,a) = x^n - x^{n-2} + a_3x^{n-3} + \cdots + a_n,\ x, a_i \in \mathbb{R} \qquad (1.1)$$

Notation 1.2. *We denote by $x_1 \leq \cdots \leq x_n$ the roots of P and by $x_1^{(k)} \leq \cdots \leq x_{n-k}^{(k)}$ the ones of $P^{(k)}$. We set $x_j^{(0)} = x_j$.*

Definition 1.1. We call *arrangement* of the roots of $P, P', \cdots, P^{(n-1)}$ the complete system of strict inequalities and equalities that hold for these roots. We assume that the roots are arranged in a string in which any two roots occupying consecutive positions are connected with a sign $<$ or $=$. An arrangement is called *non-degenerate* if there are no equalities between any two of the roots, i.e. no equalities of the form $x_i^{(j)} = x_q^{(r)}$ for any indices $(i,j) \neq (q,r)$.

The configurations of the roots of $P, P', \cdots, P^{(n-1)}$ are indicated on a figure by configuration vectors on which coinciding roots are put in square brackets. For example the configuration vector corresponding to the point $A([x_1x_2x_1^1],\ x_1^2, [x_2^1x_1^3], x_2^2, [x_3x_4x_3^1])$ means that $x_1 = x_2 = x_1^1 < x_1^2 < x_2^1 = x_1^3 < x_2^2 < x_3 = x_4 = x_3^1$.

Recall that, by applying the Rolle's theorem several times one gets for any $i < j < n$ the standard Rolle's restrictions

$$x_l^{(i)} \leq x_l^{(j)} \leq x_{l+j-i}^{(i)}.$$

And, from the properties of multiple roots, we have the obvious condition

$$\left(\left(x_k^{(i)} = x_k^{(i+1)}\right) \text{ or } \left(x_{k+1}^{(i)} = x_k^{(i+1)}\right)\right) \Rightarrow \left(x_k^{(i)} = x_k^{(i+1)} = x_{k+1}^{(i)}\right).$$

The absence of some of the arrangements is connected with the presence of overdetermined strata in any generic family of hyperbolic polynomials.

Our aim is to describe a new method to calculate these overdetermined strata in degree 4. The earlier methods are based on geometric techniques when our own is based on algebraic techniques. We transform the principal problem to the problem of resolution of a system of polynomial equations. This transformation allows the generalization of this method to degree more then4 while the geometric methods are limited to degree 5 (and maybe for some particular cases in degree 6).

A common root between two polynomials is a root of the resultant of these polynomials. We use this idea to transform the principal problem to a system of two-variables polynomials which we resolve using the Gröbner basis.

2 Overdetermined Strata (General Case)

Notation 2.1. *Denote by* $\mathrm{Pol}_n^{\mathbb{R}}$ *the space of all monic degree* n *polynomials in one variable with real coefficients. We denote by* $\mathcal{PP}_n^{\mathbb{R}}$ *the product space* $\mathrm{Pol}_n^{\mathbb{R}} \times \mathrm{Pol}_{n-1}^{\mathbb{R}} \cdots \times \mathrm{Pol}_1^{\mathbb{R}}$*. A point of* $\mathcal{PP}_n$ *is an* n*-tuple of polynomials* $(P_n, P_{n-1}, \cdots, P_1)$.

One can decompose the space $\mathcal{PP}_n$ according to the multiplicities of the roots of the different polynomials and the presence and multiplicities of their common roots. The combinatorial objects enumerating the strata should be called coloured partitions since they are partitions of C_{n+1}^2 not necessarily distinct points on $\mathbb{R}$ divided into groups of cardinalities $n, n-1, \cdots, 1$ which we can think of as having different colours (it is easy to check that this decomposition is actually a Whitney stratification).

There is a natural embedding map $\pi : \mathrm{Pol}_n^{\mathbb{R}} \hookrightarrow \mathcal{PP}_n$ sending each monic polynomial P of degree n to $(P, P'/n, P''/n(n-1), \cdots, P^{(n-1)}/n!)$.

Let λ be a coloured partition of C_{n+1}^2 coloured points, $St_\lambda \subset \mathcal{PP}_n$ be the corresponding stratum and $\pi(St_\lambda) = St_\lambda \cap \pi(\mathcal{PP}_n^{\mathbb{R}})$ be its (possibly empty) intersection with the embedded space of polynomials $\pi(\mathrm{Pol}_n^R)$. We call this intersection a stratum. Note that $\dim St_\lambda$ equals the number of parts in λ.

Definition 2.1. The stratum St_λ is called *overdetermined* if the codimension of St_λ in $\mathcal{PP}_n$ is greater than the codimension of $\pi(St_\lambda)$ in $\pi(\mathrm{Pol}_n)$ (here we assume that $\pi(St_\lambda) \neq \emptyset$). We denote by ϱ the difference between these codimensions.

Definition 2.2. An overdetermined stratum is called *non-trivial* if ϱ is due not only to the presence of the multiple roots in P and in its derivatives.

Example 2.2. *The polynomial* $(x-1)^2(x+1)^2$ *has multiple roots, but it defines a non-trivial stratum because* 0 *is a common simple root between* $P^{(1)}$ *and* $P^{(3)}$.

Remark 2.1. A polynomial P such that there are $> n-2$ equalities between roots of $P, P', \cdots, P^{(n-1)}$ belongs to an overdetermined stratum. Indeed, the latter depends on $n-2$ parameters (after the normalization $a_1 = 0,\ a_2 = -1$).

Definition 2.3. The *Gegenbauer polynomial* G_n is defined as the unique polynomial of the kind

$$x^n - x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_0$$

which is divisible by its second derivative. One can prove that it is strictly hyperbolic, and that it is odd or even together with n. The Gegenbauer polynomial $G_4 := x^4 - x^2 + \frac{5}{36}$ has by definition two roots in common with $G_4^{(2)}$ (they equal $\pm\frac{1}{\sqrt{6}}$), and $G_4^{(1)}$ has 0 as a common root with $G_4^{(3)}$. This makes three equalities between roots from the set of 10 roots of G_4, $G_4^{(1)}$, $G_4^{(2)}$ and $G_4^{(3)}$.

Remark 2.2. For all $n \geq 4$, the Gegenbauer polynomial defines an overdetermined strata in the family $P|_{a_1=0,a_2=-1}$, since it's completely defined by the condition of being divisible by its second derivative, so we get the second supplementary condition that $P^{(n-1)} = n!x$ divides all its derivatives that are odd-degree polynomials. So the quantity ϱ is equal to $(n-2)/2$.

3 Overdetermined strata for *n* = 4.

Theorem 3.1. *There are no non-trivial overdetermined strata for* $n < 4$. *For* $n = 4$ *the points* A *and* B *(see Fig. 1) define the only non-trivial overdetermined strata, and all points (excepted* A*) on the boundary of the hyperbolicity domain belong to trivial overdetermined strata.*

Proof. For $n = 2$, there is only one arrangement with at least one equality between roots; it's $[x_1 x_1^{(1)} x_2]$ that defines a trivial overdetermined stratum.

For $n = 3$, there are 4 arrangements with at least one equality between roots, they are $(x_1, x_1^{(1)}, [x_2 x_1^{(2)}], x_2^{(1)}, x_3)$, $([x_1 x_1^{(1)} x_2], x_1^{(2)}, x_2^{(1)}, x_3])$, $(x_1, x_1^{(1)}, x_1^{(2)}, [x_2 x_2^{(1)} x_3])$ and $([x_1 x_1^{(1)} x_2 x_1^{(2)} x_2^{(1)} x_3])$. The first one doesn't define an overdetermined stratum since $codim_{\mathcal{PP}_3} St_\lambda = codim_{\pi(\mathrm{Pol}_3)} \pi(St_\lambda)$. The others define trivial overdetermined strata.

For $n = 4$, at the point B we have

$$codim_{\mathcal{PP}_4} St_\lambda = 3 > codim_{\pi(\mathrm{Pol}_4)} \pi(St_\lambda) = 2,$$

so there is an overdetermined stratum.

At the point A we have $codim_{\mathcal{PP}_4} St_\lambda = 5 > codim_{\pi(\mathrm{Pol}_4)} \pi(St_\lambda) = 2$, then there exists an overdetermined stratum.

The stratum A is non-trivial because $x_2^{(1)} = x_1^{(3)}$ isn't an algebraic result of the two equalities $x_1 = x_2$ and $x_3 = x_4$, i.e. we cannot destroy $x_2^{(1)} = x_1^{(3)}$ without destroying one of $x_1 = x_2$ or $x_3 = x_4$. We can remark that for an hyperbolic polynomial if we have $x_1 = x_2$ and $x_3 = x_4$, then we should have also $x_2^{(1)} = x_1^{(3)}$.

At the point M we have a trivial overdetermined stratum since we can destroy $x_3 = x_4$ without destroying $x_2 = x_1^{(2)}$ using the change $\mathcal{P} \longmapsto \mathcal{P} + \varepsilon(x - x_2)$, $\varepsilon \lesssim 0$. By symmetry we prove that the point L defines a trivial overdetermined stratum.

At the point K there is a trivial overdetermined because we can deduce $x_2 = x_1^{(3)}$ without destroying $x_3 = x_4$ using the change $\mathcal{P} \longmapsto \mathcal{P} + \varepsilon(x - x_2)$, $\varepsilon \lesssim 0$. By symmetry we prove that the point F defines a trivial overdetermined stratum.

The point C defines a trivial overdetermined stratum because we can destroy $x_2 = x_3$ without destroying $x_2^{(1)} = x_1^{(3)}$ using the change $\mathcal{P} \longmapsto \mathcal{P} + \varepsilon$, $\varepsilon \gtrsim 0$.

The open arcs $AM, MK, KE, EC, CD, DF, FL, LA$ and the points E and D define a trivial overdetermined stratum because the value of ϱ depends only on the presence of multiple roots of P and its derivatives. There is only one case to be treated, the case where the arrangement is of the form $([x_1 x_1^{(1)} x_2 x_1^{(2)} x_2^{(1)} x_3 x_1^{(3)} x_2^{(2)} x_3^{(1)} x_4])$ which defines a trivial overdetermined stratum. □

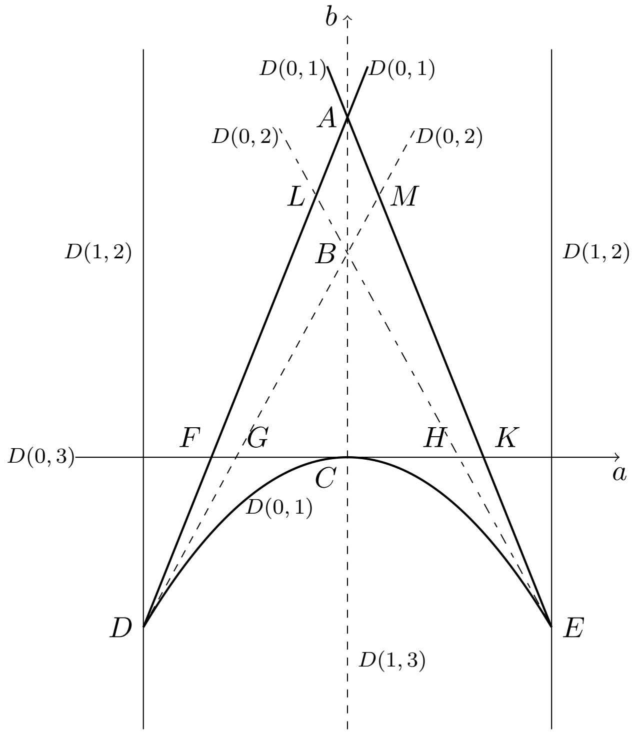


Figure 1: The hyperbolicity domain for $n = 4$. We denote by $D(i,j) = \{(a,b) \in \Pi;\ \text{Res}(P^{(i)}, P^{(j)}) = 0\}$.

4 Main Results

4.1 Resultants and subresultants

Let $P = \sum_{i=0}^{p} a_i x^i$ and $Q = \sum_{i=0}^{q} b_i x^i$ be two non-zero polynomials in one variable and of degree p and q respectively.

Definition 4.1. The *Sylvester matrix* of P and Q, denoted by $S_1(P,Q)$, is the matrix

$$S_1(P,Q) = \begin{pmatrix} a_p & \cdots & \cdots & \cdots & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & a_p & \cdots & \cdots & \cdots & \cdots & a_0 \\ b_q & \cdots & \cdots & \cdots & b_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & b_q & \cdots & \cdots & \cdots & b_0 \end{pmatrix}$$

It is a matrix of size $(p+q)\times(p+q)$. Note that its rows are

$$x^{q-1}P, \cdots, P, x^{p-1}Q, \cdots, Q$$

considered as vectors in the basis $(x^{p+q-1}, \cdots, 1)$.

The resultant of P and Q is the determinant of $S_1(P,Q)$, it is denoted by Res(P,Q).

Definition 4.2. For $k = 2, \ldots, \min(p,q)$ we define the k-th Sylvester matrix $S_k(P,Q)$ of P and Q by deleting the $q-k+2$-nd row, the last row and the last two columns of $S_{k-1}(P,Q)$. Hence $S_k(P,Q)$ is of size $(p+q+2-2k)\times(p+q+2-2k)$.

We denote by $\Delta_k(P,Q)$ the determinant of $S_k(P,Q)$. For $k=1$, Δ_1 is the resultant of P et Q.

For example if $p=4$, $q=3$ we have

$$S_1(P,Q) = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

$$S_2(P,Q) = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \\ 0 & 0 & b_3 & b_2 & b_1 \end{pmatrix} , \quad S_3(P,Q) = \begin{pmatrix} a_4 & a_3 & a_2 \\ b_3 & b_2 & b_1 \\ 0 & b_3 & b_2 \end{pmatrix} .$$

Theorem 4.1. *The two polynomials P and Q have exactly m common roots, counted with their multiplicities, if and only if $\Delta_1(P,Q) = \cdots = \Delta_m(P,Q) = 0 \neq \Delta_{m+1}(P,Q)$.*

Proof. See Proposition 4.25 in [8]. □

4.2 Algorithm

We search a and b such that $x^4 - x^2 + ax + b$ defines an overdetermined stratum. We will use the resultants and the subresultants to transform this problem to a problem of solving a system of three polynomial equations in two variables. This method is summarized in the following steps.

1. From the arrangements of the case $n = 4$, there are only 3 possible situations to discuss. See the following figures.

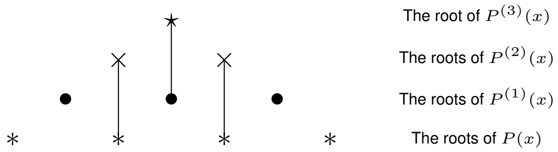


Figure 2: An eventual stratum

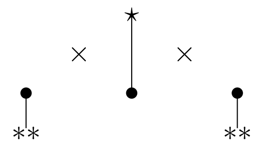


Figure 3: An eventual stratum

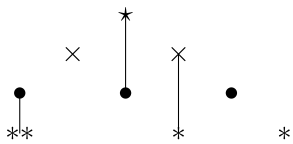


Figure 4: An eventual stratum. A straight line between two roots means a equality between them

2. • First case : there are two common roots between P and $P^{(2)}$, and another one between $P^{(1)}$ and $P^{(3)}$ so,

$$\Delta_1(P, P^{(2)}) = \Delta_2(P, P^{(2)}) = \Delta_1(P^{(1)}, P^{(3)}) = 0. \tag{4.1}$$

   • Second case : there are two common roots between P and $P^{(1)}$, and another one between $P^{(1)}$ and $P^{(3)}$ so,

$$\Delta_1(P, P^{(1)}) = \Delta_2(P, P^{(1)}) = \Delta_1(P^{(1)}, P^{(3)}) = 0. \tag{4.2}$$

   • Third case : there is only one common root between P and $P^{(1)}$, another one between P and $P^{(2)}$, another one between $P^{(1)}$ and $P^{(3)}$ so

$$\Delta_1(P, P^{(1)}) = \Delta_1(P, P^{(2)}) = \Delta_1(P^{(1)}, P^{(3)}) = 0. \tag{4.3}$$

3. • The equation (4.1) gives the following system in variables a and b:

$$\begin{cases} 400 - 5760b - 3456a^2 + 20736b^2 = 0 \\ -1728a = 0 \\ -13824a = 0 \end{cases} \tag{4.4}$$

   • The equation (4.2) gives the following system

$$\begin{cases} 16b - 128b^2 - 144ba^2 + 256b^3 + 4a^2 - 27a^4 = 0 \\ -8 + 32b + 36a^2 = 0 \\ -13824a = 0 \end{cases} \tag{4.5}$$

- The equation (4.3) gives the following system

$$\begin{cases} 16b - 128b^2 - 144ba^2 + 256b^3 + 4a^2 - 27a^4 = 0 \\ 400 - 5760b - 3456a^2 + 20736b^2 = 0 \\ -13824a = 0 \end{cases} \tag{4.6}$$

4. Using the technique of Gröbner bases, we find the equivalent systems
   - The system (4.4) is equivalent to:

$$\begin{cases} a = 0 \\ 1296b^2 - 360b + 25 = 0 \end{cases} \tag{4.7}$$

   - The system (4.5) is equivalent to:

$$\begin{cases} a = 0 \\ 4b - 1 = 0 \end{cases} \tag{4.8}$$

   - The system (4.6) is equivalent to:

$$1 = 0 \tag{4.9}$$

5. - The system (4.7) has only one solution

$$a = 0 \text{ et } b = \frac{5}{36}.$$

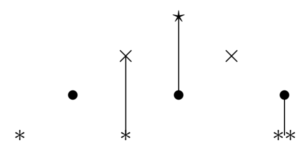
   - The system (4.8) has only one solution

$$a = 0 \text{ et } b = \frac{1}{4}.$$

   - The system (4.9) is impossible.
6. Sine the polynomials $x^4 - x^2 + \frac{5}{36}$ and $x^4 - x^2 + \frac{1}{4}$ are hyperbolic, they determine the strata in the case $n = 4$.

Remarks 4.1.

1. The previous calculations are performed using Maple.
2. The arrangement (4) give the same system ($\Delta_1(P, P^{(1)}) = \Delta_1(P, P^{(2)}) = \Delta_1(P^{(1)}, P^{(3)}) = 0$) as the one obtained in the case:



3. The polynomial that realizes the arrangement in Figure 2 is a Gegenbauer polynomial.
4. Each ideal $\mathcal{I} \neq 0$ has (for a given total order of monomials) a unique reduced Gröbner basis. (See [9]).
5. Let (S) be the following polynomial equations system:

$$S : \begin{cases} p_1(x_1, \ldots, x_n) = 0 \\ \quad \vdots \\ p_q(x_1, \ldots, x_n) = 0 \end{cases}$$

   and let $\mathcal{I}$ be the ideal generated by $p_1, \ldots, p_q$, and G be a Gröbner basis of $\mathcal{I}$. Then (S) has a solution if and only if $1 \notin G$. (See [9]).

5 Conclusion

In this article we gave a new method to calculate the overdetermined strata for $n = 4$. The methods used until now are geometric methods while our method is an algebraic method. This is a simple method that can be generalized without supplementary difficulty to degree more than 4.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Polya G, Szegö G. Problems and theorems in analysis. Springer-verlag. 1976;2.

[2] Nuij W. A note on hyperbolic polynomials. Math. Scand. 1968;23:69-72.

[3] Arnold VI. Hyperbolic polynomials and Vandermonde mappings. Funct. Anal. Appl. 1986;20(2):52-53.

[4] Arnold VI. The Newton potential of hyperbolic layers. Trudy Tbiliss Univ. 1982;232/233:23-29.

[5] Givental AB. Moments of random variables and the equivariant Morse lemma. Russian Math. Surveys. 1987;42(2):275-276 (transl. from Uspekhi Mat. Nauk 1987;42(2):221-222)).

[6] Kostov VP. On the geometric properties of Vandermonde's mapping and on the problem of moments. Proc. Roy. Soc. Edinburgh. 1989;112(3-4):203-211.

[7] Kostov VP. On the hyperbolicity domain of the polynomial $x^n + a_1x^{n-1} + \ldots + a^n$. Serdica Math. J. 1999;25(1):47-70.

[8] Basu S, Pollack R, Roy MF. Algorithms in real algebraic geometry. Second edition. Algorithms and Computation in Mathematics, 10. Springer-Verlag, Berlin, 2006. x+662 pp. ISBN: 978-3-540-33098-1; 3-540-33098-4

[9] Cox D, Little J, O'Shea D. Ideals, Varieties, and Algorithms. Undergraduate Texts in Mathematics. Springer-Verlag, 1996. second edition.

[10] Kostov VP. Discriminant sets of families of hyperbolic polynomials of degree 4 and 5. Serdica Math. J. 2002;28(2):117-152.

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