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On Existence of Solution for Higher-order Fractional Differential Inclusions with Anti-periodic Type Boundary Conditions

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Abstract

This paper investigates the existence of solutions for fractional differential inclusions of order $q \in (4, 5)$ with anti-periodic type boundary conditions by means of some standard fixed point theorems for inclusions. The existence results are established for convex as well as the non-convex multivalued maps. Some illustrative examples are introduced to explain the applicability of the theory.

Keywords: Existence; Fractional differential inclusions; anti-periodic boundary condition; fixed point theorems; multivalued maps.

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1 Introduction

The topic of fractional differential equations and inclusions has recently emerged as a popular field of research due to its extensive development and applications in several disciplines such as physics, mechanics, chemistry, engineering, etc. (see [1], [2], [3], [4], [5], [6], [7], and references therein). The fact that using the fractional-order models instead of integer-order model is due to more realistic

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in description of many physical phenomenons. The investigation of existence problems of fractional differential equations in general is considered as a priority for going forward in such applications (see [8], [9], [10], [11],[12]). Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes ([6], [13]) and have recently received considerable attention. For examples and details of anti-periodic boundary conditions (see [14]-[20], [13], [21], [22] and the references therein). Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors (see [23], [24] and the references therein). For some recent works on differential inclusions of fractional order, we refer the reader to the references ([25], [18], [19]). In this paper, we discuss some existence results for anti-periodic boundary value problems of differential inclusions of fractional order $q \in (4, 5]$.

Precisely, we consider the following problem:

$$\begin{cases} {}^cD^q x\left(t\right) \in F\left(t, x\left(t\right)\right), t \in J = [0, T], T > 0, q \in (4, 5] \\ x^{(k)}\left(0\right) = -x^{(k)}\left(T\right), k = 0, 1, 2, 3, 4. \end{cases} \tag{1.1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, and ${}^cD^q$ denotes the Caputo fractional derivative of order q which is generally defined by

$${}^cD^q x(t) = I^{n-q} x^{(n)}(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} x^{(n)}(s) ds, n-1 < q < n,$$

where $n = [q] + 1$, and $[q]$ denotes the integer part of the real number q.

This paper is organized as follows: In Section 2, we introduce some well-known results in multivalued analysis. The main results of existence theorems will be given in Section 3. Finally, we give some illustrative examples to explain the theorems.

2 Preliminaries

We recall in this section some facts from multivalued mapping analysis (see [26], [27], [28]) that needed for the results in the sequel.

Definition 2.1. For a normed space $(X, \|\cdot\|)$, let
$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ is closed$\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y$ is bounded$\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact$\}$, and
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) :\ Y$ is compact and convex$\}$.

Definition 2.2. Let $F : X \to \mathcal{P}(X)$ be a multivalued map.
(i) F is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$.
(ii) F is bounded on bounded sets if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in X for all $B \in P_b(X)$.
(iii) F is an upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood $N_0$of $x_0$such that $F(N_0) \subseteq N$.
(iv) F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$.
(v) F has a fixed point if there is $x \in X$ such that $x \in F(x)$.
(vi) If F is completely continuous with nonempty compact values, then F is u.s.c if and only if F has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

The fixed point set of the multivalued operator F will be denoted by $Fix\ F$.

Definition 2.3. A multivalued map $F : J \to \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, F(t)) = inf\{|y - z| : z \in F(t)\}$$

is measurable.

Let $L^1(J, \mathbb{R})$ be the Banach space of all measurable functions $x : J \to \mathbb{R}$ which are Lebesgue integrable endowed with the norm $\|x\|_{L^1} = \int_0^T |x(t)|dt$, and $C(J, \mathbb{R})$ denotes the Banach space of all real valued continuous functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in J\}$.

Definition 2.4. .A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if (i) $t \to F(t, x)$ is measurable for each $x \in \mathbb{R}$, (ii) $x \to F(t, x)$ is upper semi-continuous for almost all $t \in J$. Further a Carathéodory function F is called L^1-Carathéodory If for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in J$.

Definition 2.5. Let Y be a Banach space, Z a nonempty closed subset of Y. The multivalued operator $F : Z \to \mathcal{P}(Y)$ is said to be lower semi-continuous (l.s.c.) if the set $\{z \in Z : F(z) \cap B \neq \phi\}$ is open for any open set B in Y .

Definition 2.6. Let A be a subset of $J \times \mathbb{R}$. A is said to be $\mathcal{L} \otimes \mathcal{B}$-measurable if A belongs to the σ-algebra generated by all sets of the form $L \times B$, where L is Lebesgue measurable in J and B is Borel measurable in $\mathbb{R}$.

Definition 2.7. A subset A of $L^1(J, \mathbb{R})$ is decomposable if for all $u, v \in A$ and measurable sets $I \subset J$, the function $u\chi_I + v\chi_{J-I} \in A$, where χ_I stands for the characteristic function of I.

Definition 2.8. If $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with nonempty compact values and $u \in C(J, \mathbb{R})$, then the set of selections of $F(.,.)$, denoted by $S_{F,u}$, is of lower semi-continuous type if

$$S_{F,u} = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, u(t)) \text{ for a.e. } t \in J\}$$

is lower semi-continuous with nonempty closed and decomposable values .

Definition 2.9. Let (X, d) be a metric space associated with the metric d. The Pompeiu–Hausdorff distance of the closed subsets $A,\ B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},$$

where $d^*(A, B) = \sup\{d(a, B) : a \in A\}$, and $d(x, B) = \inf_{y \in B} d(x, y)$.

Definition 2.10. A multivalued operator F on X with nonempty values in X is called (a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_H(F(x), F(y)) \leq \gamma d(x, y), \text{for each } x, y \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemmas will be used in what follows.

Lemma 2.1. *([29])Let X be a Banach space. Let $F : J \times X \to P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let H be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\begin{array}{rcl} H \circ S_F & : & C(J,X) \to P_{cp,c}(C(J,X)), \\ x & \to & (H \circ S_F)(x) = H(S_{F,x}) \end{array}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

We close these preliminaries by introducing the following two fixed point theorems.

Lemma 2.2. *([30])Let Y be a separable metric space and let $F : Y \to \mathcal{P}(L^1(J, \mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $F(.)$ has a continuous selection, i.e., there exists a continuous mapping (single valued) $f : Y \to L^1(J, \mathbb{R})$ such that $f(y) \in F(y)$ for every $y \in Y$.*

Lemma 2.3. *([31]) Let (X, d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then $FixF \neq \phi$.*

3 Existence Results

In this section, three existence results of problem (1.1) are presented. The first one concerns the convex valued case, and the others are related to the nonconvex valued case.

Before starting the first result, we recall an equivalent integral form of the corresponding single value problem of (1.1).

Lemma 3.1. *([20])For any $y \in C(J, \mathbb{R})$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), t \in J, 4 < q \leq 5, \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, 3, 4, \end{cases}$$

is

$$x(t) = \int_0^T G(t,s)\, y(s)\, ds,$$

where $G(t, s)$ is Green's function given by

$$G(t,s) = \begin{cases} \frac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} + \\ \frac{\left(6t^2T - 4t^3 - T^3\right)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{\left(3t^3T - t^4 - tT^3\right)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 < s < t < T, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q--1)} + \frac{t(T-t)(T-s)^{q-2}}{4\Gamma(q-2)} + \\ \frac{\left(6t^2T - 4t^3 - T^3\right)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{\left(3t^3T - t^4 - tT^3\right)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 < t < s < T. \end{cases}$$

Observe that

$$\begin{array}{l} (T-t)(T-s)^{q-5} \leq (T-t)^{q-4}, t < s, \\ (T-t)(T-s)^{q-5} \leq (T-s)^{q-4}, t \geq s. \end{array}$$

The main results are based on the following fixed point theorems.

Theorem 3.2. *[32](Nonlinear alternative of Leray-Schauder type) Let X be a Banach space, $\mathfrak{X}$ be a closed convex subset of X, $\mathfrak{U}$ be an open subset of $\mathfrak{X}$ with $0 \in \mathfrak{U}$. Suppose that $F : \overline{\mathfrak{U}} \to P_{cp,c}(\mathfrak{X})$ is an upper semicontinuous compact map. Then either F has a fixed point in $\overline{\mathfrak{U}}$ or there are $\mathfrak{x} \in \partial\mathfrak{U}$ and $\lambda \in (0, 1)$ such that $\mathfrak{x} \in \lambda F(\mathfrak{x})$.*

Theorem 3.3. *[32](Covitz and Nadler) Let (X, d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then F has a fixed point.*

Before going on, using Lemma 3.1, we convert the problem to be suitable form for using the above fixed point theorems, hence we let H acting on $S_{F,x}$ as

$$\begin{aligned} H(f)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\,ds \\ &+ \frac{\left(6t^2T - 4t^3 - T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)\,ds \\ &+ \frac{\left(3t^3T - t^4 - tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s)\,ds, \end{aligned} \tag{3.1}$$

for $t \in J$. Using the convexity property of the multivalued map, we can prove the first result.

Theorem 3.4. *Assume that*

(H1) $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has convex values.*

(H2) *There exists a continuous nondecreasing function $\alpha : [0, \infty) \to (0, \infty)$ and a function $p \in L^1(J, \mathbb{R}^+)$ with $\|p\|_{L^1} > 0$, such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le p(t)\alpha(\|x\|) \text{ for each } (t,x) \in J \times \mathbb{R}.$$

(H3) *There exists a number $M > 0$ such that*

$$\frac{M}{\gamma\alpha(M)\,\|p\|_{L^1}} > 1,$$

where

$$\gamma = \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)(10 + 35(q-1) + 4(q-1)^3)}{48}\right).$$

Then the boundary value problem (1.1) has at least one solution on J.

Proof. Define an operator $\Upsilon(x) = \{h_f \in C(J,\mathbb{R}) : h_f = H(f) \text{ for } f \in S_{F,x}\}$. We show that Υ satisfies the assumptions of the nonlinear alternative of the Leray-Schauder type. The proof consists of several steps.

Step One: We show that $\Upsilon(x)$ is convex for each $x \in C(J,\mathbb{R})$. For that, let $h_{f_1}, h_{f_2} \in \Upsilon(x)$, such that h_{f_1}, and h_{f_2} satisfying (3.1) for $f_1,\ f_2 \in S_{F,x}$. Therefore, if $0 \le \omega \le 1$ and $t \in J$, we have

$$\begin{aligned}
&[\omega h_{f_1} + (1-\omega)h_{f_2}](t) \\
=\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
& -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
& +\frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
& +\frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
& +\frac{\left(6t^2T - 4t^3 - T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
& +\frac{\left(3t^3T - t^4 - tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}[\omega f_1(s) + (1-\omega) f_2(s)]ds.
\end{aligned}$$

Since F has convex values implies $S_{F,x}$ is convex, and then $\omega h_{f_1} + (1-\omega)h_{f_2} \in \Upsilon(x)$.

Step Two: We show that $\Upsilon(x)$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$. For a positive number r, let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C(J,\mathbb{R})$. Then, $h_f \in \Upsilon(x),\ x \in B_r$, implies h_f satisfies (3.1) for $f \in S_{F,x}$, and

$$\begin{aligned}
|h_f(t)| \le\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s)|ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}|f(s)|ds \\
& +\frac{|T-2t|}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s)|ds + \frac{|t(T-t)|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s)|ds \\
& +\frac{|6t^2T - 4t^3 - T^3|}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}|f(s)|ds \\
& +\frac{|3t^3T - t^4 - tT^3|}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}|f(s)|ds.
\end{aligned}$$

Hence

$$\|h_f\| \le \frac{\alpha(M)T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)(10+35(q-1)+4(q-1)^3)}{48}\right)\int_0^T |p(s)|ds.$$

Step Three: We show that Υ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$ where B_r is a bounded set of $C(J,\mathbb{R})$. In view of (H_3), for each $h_f \in \Upsilon(x)$, we obtain

$$|h_f(t_2) - h_f(t_1)| \le \int_0^{t_2} \frac{|(t_2-s)^{q-1} - (t_1-s)^{q-1}|}{\Gamma(q)}|f(s)|ds + \int_{t_1}^{t_2} \frac{|(t_2-s)^{q-1}|}{\Gamma(q)}|f(s)|ds$$

$$+\frac{|t_2-t_1|}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\left|f(s)\right|ds+\frac{|(t_2-t_1)(T-t_2-t_1)|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\left|f(s)\right|ds$$
$$+\frac{|(t_2-t_1)\left[3T(t_2+t_1)-2(t_2^2+t_2t_1+t_1^2)\right]|}{24}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\left|f(s)\right|ds$$
$$+\frac{|(t_2-t_1)[2(t_2^2+t_1t_2+t_1^2)T-(t_2+t_1)(t_2^2+t_1^2)-T^3]|}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}|f(s)|ds$$
$$\leq \int_0^{t_2}\frac{\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|}{\Gamma(q)}\left|p(s)\alpha(M)\right|ds+\int_{t_1}^{t_2}\frac{|(t_2-s)^{q-1}|}{\Gamma(q)}\left|p(s)\alpha(M)\right|ds$$
$$+\frac{|t_2-t_1|}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\left|p(s)\alpha(M)\right|ds$$
$$+\frac{|(t_2-t_1)[T-t_2-t_1]|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\left|p(s)\alpha(M)\right|ds$$
$$+\frac{|(t_2-t_1)[3T(t_2+t_1)-2(t_2^2+t_2t_1+t_1^2)]|}{24}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}|p(s)\alpha(M)|ds$$
$$+\frac{|(t_2-t_1)[2(t_2^2+t_2t_1+t_1^2)T-(t_2+t_1)(t_2^2+t_1^2)-T^3]|}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}|p(s)\alpha(M)|ds.$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. The operator Υ satisfies the above three assumptions, it follows by the Arzelá–Ascoli theorem that $\Upsilon : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Step Four: We show that Υ has a closed graph. Let $x_n \to x_*$, $h_{f_n} \in \Upsilon(x_n)$ and $h_{f_n} \to h_{f_*}$. Then we need to show that $h_{f_*} \in \Upsilon(x_*)$ i.e h_{f_*} satisfies (3.1) for $f_* \in S_{F,x_*}$. Since $h_{f_n} \in \Upsilon(x_n)$, then it satisfies (3.1) for $f_n \in S_{F,x_n}$ and $t \in J$. Now, the operator $H : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$, defined by

$$f \to (H \circ S_{F,x})(f) = H(f)$$

and the last given in (3.1), is a continuous linear operator. Observe that $\|h_{f_n} - h_{f_*}\| = \|h_{f_n - f_*}\| \to 0$, as $n \to \infty$. Thus, it follows by Lemma 2.1 that $H \circ S_{F,x}$ is a closed graph operator. Further, we have $h_{f_n}(t) \in H(S_{F,x})$, since $x_n \to x_*$, we have h_{f_*} satisfies (3.1) for some $f_* \in S_{F,x_*}$.

Step Five: Finally, we discuss a priori bounds on solutions. Let x_f be a solution of (1.1). Then x_f satisfies (3.1) for $f \in S_{F,x_f}$. Using $(H2)$, we obtain

$$\begin{aligned}|x_f(t)| &\leq \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(10+35(q-1)+4(q-1)^3}{48}\right)\int_0^T p(s)ds\\ &\leq \gamma\alpha\left(\|x_f\|\right)\int_0^T p(s)ds.\end{aligned}$$

Consequently, we have

$$\frac{\|x_f\|}{\gamma\alpha\left(\|x_f\|\right)\|p\|_{L^1}} \leq 1.$$

In view of $(H3)$, there exists M such that $\|x_f\| \neq M$. Let us set $\mathfrak{U} = \{x \in C(J,\mathbb{R}) : \|x\| < M+1\}$. Note that the operator $\Upsilon : \overline{\mathfrak{U}} \to \mathcal{P}(C(J,\mathbb{R}))$ is an upper semi-continuous and completely continuous. The choice of $\mathfrak{U}$, implies there is no $x \in \partial\mathfrak{U}$ such that $x \in \lambda\Upsilon(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of the Leray-Schauder type (3.2), we deduce that Υ has a fixed point $x \in \overline{\mathfrak{U}}$ which is a solution of problem (1.1). This completes the proof. □

As a next result, we study the case when F is not necessarily convex valued. Since the convexity may be replaced by decomposability, then the next result is based on the nonlinear alternative of the Leray–Schauder type together with the selection theorem of Bressan and Colombo ([30]) for lower semi-continuous maps with decomposable values .

Theorem 3.5. *Assume that $(H2)$, $(H3)$ and the following conditions hold*

(H4) $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
(a) $(t, x) \to F(t, x)$ *is* $\mathcal{L} \otimes \mathcal{B}$*-measurable,*
(b) $x \to F(t, x)$ *is* $l.s.c$ *for each* $t \in J$.

(H5) *For each* $\delta > 0$*, there exists* $\varphi_\delta \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\delta(t) \text{ for all } \|x\| \leq \delta \text{ and for a.e. } t \in J.$$

Then the boundary value problem (1.1) has at least one solution on J.

Proof. It follows from $(H4)$ and $(H5)$ that F is of $l.s.c.$ type and has nonempty closed and decomposable values. Then from Lemma 2.2, there exists a continuous function $f : C(J, \mathbb{R}) \to L^1(J, \mathbb{R})$ such that $f(x)(t) \in F(x)$ for all $x \in C(J, \mathbb{R})$, and a.e. $t \in J$. Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x)(t), t \in J, q \in (4, 5] \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, 3, 4. \end{cases} \tag{3.2}$$

Observe that if $x \in C(J, \mathbb{R})$ is a solution of (3.2), then x is a solution to problem (1.1). In order to transform problem (3.2) into a fixed point problem, we define the operator $\Pi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as

$$\Pi(x)(t) = H(f(x))(t).$$

It can easily be shown that Π is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.4. So we omit it. This completes the proof. ☐

Now we prove the existence of solutions for problem (1.1) with a nonconvex-valued right hand side by applying a fixed point theorem for a multivalued map due to Theorem (3.3).

Theorem 3.6. *Assume that the following conditions hold:*

(H6) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x) : J \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

(H7) $d_H(F(t,x), F(t, y)) \leq z(t)|x - y|$ *for almost all* $t \in J$ *and* $x, y \in \mathbb{R}$ *with* $z \in L^1(J, \mathbb{R}^+)$ *and* $d_H(0, F(t, 0)) \leq z(t)$ *for almost all* $t \in J$.

Then the boundary value problem (1.1) has at least one solution on J *if*

$$\frac{T^{q-1}}{\Gamma(q)} \left(\frac{3}{2} + \frac{(q-1)(10 + 35(q-1) + 4(q-1)^3)}{48} \right) \|z\|_{L^1} < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J, \mathbb{R})$ by assumption $(H6)$, so F has a measurable selection (see [33], Theorem III.6]). Now we show that the operator Υ (defined as in the proof of Theorem 3.4) satisfies the requirements of Lemma 2.3.

Step One: We show that $\Upsilon(x) \in P_{cl}((C(J, \mathbb{R}))$ for each $x \in C(J, \mathbb{R})$. Let $(u_n)_{n \geq 0} \in \Upsilon(x)$ be such that $u_n \to u$ in $C(J, \mathbb{R})$. Then $u \in C(J, \mathbb{R})$ and u_n satisfies (see equation (3.1)) $u_n(t) = H(v_n)(t)$ for some $v_n \in S_{F,x}$, $t \in J$. As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1(J, \mathbb{R})$. Thus, u satisfies $u(t) = H(v)(t)$ for $v \in S_{F,x}$ and $t \in J$. Hence, $u \in \Upsilon(x)$.

Step Two: We show that there exists $0 < \beta < 1$ such that

$$d_H(\Upsilon(x), \Upsilon(y)) \leq \beta \|x - y\|$$

for each $x, y \in C(J, \mathbb{R})$. Let $x, y \in C(J, \mathbb{R})$ and $u_1 \in \Upsilon(x)$. Then u_1 satisfies $u_1(t) = H(v_1)(t)$ for $v_1(t) \in F(t, x(t))$ and $t \in J$. By $(H7)$, we have

$$d_H(F(t,x), F(t,y)) \leq z(t)|x(t) - y(t)|,$$

for almost all $t \in J$. So, there exists $w \in F(t, y(t))$ such that

$$|v_1(t) - w| \leq z(t)|x(t) - y(t)|, \ t \in J.$$

Define the multivalued map $V : J \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq z(t)\,|x(t) - y(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t, y(t))$ is measurable ([33], Proposition III.4), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, y(t))$ and for each $t \in J$, we have

$$|v_1(t) - v_2(t)| \leq z(t)|x(t) - y(t)|, \text{ for a.e. } t \in J.$$

Let u_2 satisfying $u_2(t) = H(v_2)(t)$. Thus, for each $t \in J$, it follows that

$$\begin{aligned}
& |u_1(t) - u_2(t)| \\
\leq\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|\,ds \\
& + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|\,ds \\
& + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |v_1(s) - v_2(s)|\,ds \\
& + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |v_1(s) - v_2(s)|\,ds \\
& + \frac{\left|6t^2T - 4t^3 - T^3\right|}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |v_1(s) - v_2(s)|\,ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{\left|3t^3T - t^4 - tT^3\right|}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} |v_1(s) - v_2(s)|\,ds \\
\leq\ & \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)(10 + 35(q-1) + 4(q-1)^3)}{48}\right)\int_0^T z(s)\,\|x-y\|ds.
\end{aligned}$$

Hence,

$$\|h_1(t) - h_2(t)\| \leq \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)(10 + 35(q-1) + 4(q-1)^3)}{48}\right)\|z\|_{L^1}\|x-y\|.$$

Analogously, interchanging the roles of x and y, we obtain

$$\begin{aligned}
d_H(\Upsilon(x), \Upsilon(y)) &\leq \beta\|x-y\| \\
&\leq \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)(10 + 35(q-1) + 4(q-1)^3)}{48}\right)\|z\|_{L^1}\|x-y\|.
\end{aligned}$$

Since $\frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)(10+35(q-1)+4(q-1)^3)}{48}\right)\|z\|_{L^1} < 1$, then Υ is a contraction. It follows by Lemma 2.3 that Υ has a fixed point x which is a solution of (1.1). This completes the proof. □

The new existence results for a class of fifth-order nonlinear differential inclusions with anti-periodic boundary conditions follow as a special case by taking $q = 5$ in the results of this section.

Example 3.7. *Consider the following fractional differential inclusion*

$$\begin{cases} {}^cD^{4.2}x(t) \in F(t, x(t)), t \in [0,1], \\ x^{(k)}(0) = -x^{(k)}(1), k = 0,1,2,3,4. \end{cases} \tag{3.3}$$

where $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a multivalued map given by*

$$F(t,x) = \left\{ y \in \mathbb{R} : e^{-|x|} + \sin t + t^2 \le y \le 2 + \frac{|x|}{1+x^2} + t^3 \right\}.$$

It is clear that F *is Carathéodory and has convex values satisfying*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le 4 \text{ for each } (t,x) \in J \times \mathbb{R},$$

with $p(t) = 1,$ *and* $\alpha(\|x\|) = 4.$ *Furthermore, let* M *be any number satisfying*

$$\begin{aligned} M &> \frac{T^{q-1}\alpha(M)\,\|p\|_{L^1}}{\Gamma(q)} \left(\frac{3}{2} + \frac{(q-1)(10+35(q-1)+4(q-1)^3)}{48} \right) \\ &> 3.49. \end{aligned}$$

Clearly, all the conditions of Theorem 3.4 are satisfied. So there exists at least one solution of problem (3.3) on $[0,1]$.

Example 3.8. *Consider the following fractional differential inclusion*

$$\begin{cases} {}^cD^{\frac{19}{4}}x(t) \in F(t, x(t)), t \in [0,1], \\ x^{(k)}(0) = -x^{(k)}(1), k = 0,1,2,3,4. \end{cases} \tag{3.4}$$

where $F : [0,1] \times [0, \frac{\pi}{2}] \to \mathbb{R}^+$ *is a multivalued map given by*

$$F(t,x) = \left[0, \frac{\sin x}{(2+t)^4} \right].$$

Now

$$\sup\{|y| : y \in F(t,x)\} \le \frac{\sin x}{(2+t)^4} \le \frac{1}{16} \text{ for each } (t,x) \in [0,1] \times [0, \frac{\pi}{2}],$$

and

$$d_H(F(t,x), F(t,y)) \le \frac{1}{(2+t)^4}|x-y|.$$

Here $z(t) = \frac{1}{(2+t)^4}$, *with* $\|z\|_{L^1} = 0.017,$ *and*

$$\frac{T^{q-1}}{\Gamma(q)} \left(\frac{3}{2} + \frac{(q-1)(10+35(q-1)+4(q-1)^3)}{48} \right) \|z\|_{L^1} \le 0.009 < 1.$$

The compactness of F *together with the above calculations lead to the existence of solution of the problem (3.4) by Theorem 3.6.*

4 Conclusion

The fractional differential inclusion (1.1) of order $q \in (4, 5]$ with anti-periodic type boundary condition is considered by means of some standard fixed point theorems for inclusions. The existence results are established for convex as well as the non-convex multivalued maps by obtaining sufficient conditions for each case. The fact that the problem is a generalization of lower fractional order inclusion problems. The more generalization for arbitrary fractional order inclusion problem is still open problem for the researchers.

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Competing Interests

The authors declare that no competing interests exist.

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