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A Remark on the Theorem of Ishikawa

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Abstract

Given a Lipschitz pseudocontractive mapping T from a closed convex and bounded subset K of a real Hilbert space H onto itself, and an arbitrary $x_1 \in K$, a *Krasnolselskii-type* sequence defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda T y_n,$$
$$y_n = (1-\lambda)x_n + \lambda T x_n$$

is proved to be an approximate fixed point sequence of T, for a suitable $\lambda \in (0,1)$. Under some suitable compactness assumptions on K or on T, the sequence converges strongly to a fixed point of T. The algorithm is simple and natural, and the theorems presented here improve the theorem of Ishikawa [1] and other similar results in the literature.

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1 Introduction

Let H be a real Hilbert space and K a closed convex and bounded subset of H. A mapping $T : K \to K$ is said to be

- *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - y) - (Tx - Ty)\|^2$$

  for all $x, y \in K$ and

- *Lipschitzian* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

Pseudocontractive mappings are generalizations of nonexpansive mappings and have been studied extensively, for example, by Browder[2], Browder and Petryshyn [3], Kirk [4], Martinet [5], Xu [6] and a host of other authors. Closely connected with pseudocontractive mapping are the accretive mappings(i.e a mapping T satisfying the condition that $I - T$ is pseudocontractive). For datails on accretive mappings one may consult, for example, Chidume and Zegeye [7], Chugh [8].

In 1974, in the setting where T is Lipschitzian and pseudocontractive and K compact, Ishikawa [1] introduced an iteration process(now known as the Ishikawa process) and proved strong convergence theorems to a fixed point of T as follows:

Theorem 1.1. *[1] If E is a convex compact subset of a Hilbert space H, T is a Lipschitzian pseudo-contractive map from E into itself and x_1 is any point in E, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer n by*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n, \tag{1.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences of positive numbers that satisfy the following three conditions:

$$0 \leq \alpha_n \leq \beta_n \leq 1 \text{ for all positive integers } n,$$
$$\lim_{n\to} \beta_n = 0,$$
$$\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty.$$

In general, the Ishikawa algorithm (1.1) is computationally expensive and it is also not so easy to find sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that satisfy the system above.

For several years, it was a problem of interest to know whether the Mann iteration process defined by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 1,$$

for an appropriate $\{\alpha_n\} \subseteq (0, 1)$ would, *always*(in a similar setting), converge strongly to a fixed point of this class of mappings. Certainly, the Mann iteration process is less computationally involved than the Ishikawa process. Moreover, the order of convergence of the Mann process is $\frac{1}{n}$ whereas that of Ishikawa is $\frac{1}{\sqrt{n}}$. Thus, if the Mann process converges, then it is more desirable than the Ishikawa process.

Indeed, strong convergence of the Ishikawa and Mann iteration processes to a fixed point of T have been established(see, for example, Browder [2]), even in normed linear spaces, in the case where T belongs to a proper subclass of Lipschitz pseudocontractive mapping called strictly pseudocontractive mappings.

In 2001, Chidume and Mutangadura [9] gave an example to show that, for a Lipschitzian pseudo-contractive mapping T defined on a real Hilbert space, a Mann iteration process may fail to converge to a fixed point of T, even when the set K is compact and the fixed point of T is unique. Thus the problem was resolved in the negative.

Though the Mann sequence does not always converge for this class of mappings, nevertheless, it was a problem of interest to find an iteration process which is more efficient and readily applicable than the Ishikawa process and which will, *always*, converge to a fixed point of a Lipschitzian pseudo-contractive mapping.

It is our purpose in this paper to establish that a certain meanvalue sequence is, *always*, an approximate fixed point sequence of T. Moreover, if we further assume that T is *hemicompact*(or, in particular, compact as it is in [1]), then the sequence converges strongly to a fixed point of T. Our theorems improve the result of Ishikawa and complement several known result in the literature.(see, e.g., [10], [11]).

2 Preliminaries

We introduce in this section some definitions, notations and results which will be needed in proving our main results:

(i) $x_n \to x$: $\{x_n\}$ converges strongly to x as $n \to \infty$.

(ii) H: a real Hilbert space with an induced norm $\|.\|$.

(iii) $F(T) := \{x \in K : x = Tx\}$.

We recall the following proposition.

Definition 2.1. A map $T : K \to K$ is said to be *hemicompact* if, for any sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence, say, $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in F(T)$.

Note that if K is compact, then every mapping $T : K \to K$ is hemicompact.

The following lemma will also be used in the sequel.

Lemma 2.1. *Let H be a Hilbert space. Then the following identity holds:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.1}$$

for all $\lambda \in (0, 1)$ and $x, y \in H$

Lemma 2.2. *([6]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \sigma_n, \;\; n \geq 0,$$

such that $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$. Then, $\lim a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence that converges to 0, then a_n converges to 0 as $n \to \infty$.

3 Main Results

We prove the following theorems.

Theorem 3.1. *Let H be a Hilbert space, $K \subseteq H$ be a nonempty, closed and convex. Let T be a Lipschitzian and pseudocontractive self-map of K, with Lipschitz constant $L > 0$, such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by $x_1 \in K$ and*

$$x_{n+1} = (1-\lambda)x_n + \lambda T y_n, \tag{3.1}$$

$$y_n = (1-\lambda)x_n + \lambda T x_n, \tag{3.2}$$

where $\lambda \in (0, L^{-2}[\sqrt{1+L^2} - 1])$. Then, for each $p \in F(T)$, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Let $p \in F(T)$. Using Lemma 2.1, and following a procedure similar to that of Ishikawa [1], we have

$$\|x_{n+1} - p\|^2 = \|(1-\lambda)(x_n - p) + \lambda(Ty_n - p)\|^2$$

$$= (1-\lambda)\|x_n - p\|^2 + \lambda\|Ty_n - p\|^2 - \lambda(1-\lambda)\|x_n - Ty_n\|^2, \tag{3.3}$$

$$\|Ty_n - p\|^2 = \|Ty_n - Tp\|^2 \leq \|y_n - p\|^2 + \|y_n - Ty_n\|^2, \tag{3.4}$$

$$\|y_n - p\|^2 = \|(1-\lambda)(x_n - p) + \lambda(Tx_n - p)\|^2,$$

$$= (1-\lambda)\|x_n - p\|^2 + \lambda\|Tx_n - p\|^2 - \lambda(1-\lambda)\|x_n - Tx_n\|^2, \tag{3.5}$$

$$\|y_n - Ty_n\|^2 = \|(1-\lambda)(x_n - Ty_n) + \lambda(Tx_n - Ty_n)\|^2,$$

$$= (1-\lambda)\|x_n - Ty_n\|^2 + \lambda\|Tx_n - Ty_n\|^2 - \lambda(1-\lambda)\|x_n - Tx_n\|^2, \tag{3.6}$$

and

$$\|Tx_n - p\|^2 = \|Tx_n - Tp\|^2 \leq \|x_n - p\|^2 + \|x_n - Tx_n\|^2 \tag{3.7}$$

Substituting (3.4)-(3.7) into (3.3), we have

$$\|x_{n+1} - p\|^2 = (1-\lambda)\|x_n - p\|^2 + \lambda\|Ty_n - p\|^2 - \lambda(1-\lambda)\|x_n - Ty_n\|^2$$

$$\leq (1-\lambda)\|x_n - p\|^2 + \lambda[\|y_n - p\|^2 + \|y_n - Ty_n\|^2] - \lambda(1-\lambda)\|x_n - Ty_n\|^2. \tag{3.8}$$

Thus

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\lambda)\|x_n - p\|^2 + \lambda[(1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda\|x_n - Tx_n\|^2 \\ &\quad - \lambda(1-\lambda)\|x_n - Tx_n\|^2 + (1-\lambda)\|x_n - Ty_n\|^2 + \lambda\|Tx_n - Ty_n\|^2 \\ &\quad - \lambda(1-\lambda)\|x_n - Tx_n\|^2] - \lambda(1-\lambda)\|x_n - Ty_n\|^2 \\ &= \|x_n - p\|^2 + \lambda^2\|x_n - Tx_n\|^2 - 2\lambda^2(1-\lambda)\|x_n - Tx_n\|^2 + \lambda^2\|Tx_n - Ty_n\|^2\end{aligned}$$

$$\leq \|x_n - p\|^2 - \lambda^2(1 - 2\lambda - \lambda^2 L^2)\|x_n - Tx_n\|^2. \quad (3.9)$$

From (3.9), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|, \quad (3.10)$$

and

$$\lambda^2(1 - 2\lambda - \lambda^2 L^2)\|x_n - Tx_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \quad (3.11)$$

Using (3.10) and Lemma 2.2, we have that

$$\lim_{n\to\infty} \|x_n - p\|$$

exists. Moreover, $1 - 2\lambda - \lambda^2 L^2 > 0 \quad \Leftrightarrow \quad |\lambda + \frac{1}{L^2}| < L^{-2}\sqrt{L^2 + 1}$. Therefore, since $0 < \lambda \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$, we have $1 - 2\lambda - \lambda^2 L^2 > 0$. Taking limits on both sides of (3.11), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

□

We now prove the following corollaries

Corollary 3.2. *Let H be a Hilbert space, $K \subseteq H$ be a nonempty, closed and convex. Let T be a hemicompact, Lipschitzian and pseudocontractive self-map of K, with Lipschitz constant $L > 0$, such that $F(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a fixed point of T.*

Proof. Using Theorem 3.1, we obtain $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since T is hemicompact, we have a subsequence of $\{x_n\}$, say $\{x_{n_k}\}$, which converges strongly to some $q \in K$. Since $\{x_n\}$ is nonincreasing, we have $x_n \to q$ as $n \to \infty$. Therefore

$$\begin{aligned} \|q - Tq\| &\leq \|q - x_n\| + \|x_n - Tx\| + \|Tx_n - Tq\| \\ &\leq \|q - x_n\| + \|x_n - Tx\| + L\|x_n - q\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

Thus $\{x_n\}$ converges strongly to $q \in F(T)$. □

Corollary 3.3. *Let H be a Hilbert space, $K \subseteq H$ be a nonempty, compact and convex. Let T be a Lipschitzian and pseudocontractive self-map of K, with Lipschitz constant $L > 0$. Then, the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a fixed point of T.*

Proof. Since K is nonempty, convex and compact and T continuous, Schauder's fixed point theory guarantees that $F(T)$ is nonempty. Moreover, T is hemicompact since $T(K)$ is compact. Thus by Corollary 3.2, $\{x_n\}$ converges strongly to a fixed point of T. □

Corollary 3.4. *Let H be a Hilbert space, $K \subseteq H$ be a nonempty, closed and convex. Let T be a Lipschitzian and pseudocontractive self-map of K, with Lipschitz constant $L > 0$, such that $F(T) \neq \emptyset$. Assume that the interior of $F(T)$ is nonempty. Then, the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a fixed point of T.*

Proof. Using Theorem 3.1, we have that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Now the following identity is true in Hilbert spaces.

$$\|x_n - p\|^2 = \|x_{n+1} - x_n\|^2 + \|x_{n+1} - p\|^2 + 2\langle x_n - x_{n+1}, x_{n+1} - p\rangle$$

and therefore

$$\langle x_n - x_{n+1}, x_{n+1} - p\rangle + \frac{1}{2}\|x_{n+1} - x_n\|^2 = \frac{1}{2}\Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big) \tag{3.12}$$

Since $intF(T) \neq \emptyset$, let $x^* \in F(T)$ and $r > 0$ such that $x^* + rh \in F(T)$ for all h with $\|h\| < 1$. Then, by inequalities (3.12) and (3.10), we have

$$\langle x_n - x_{n+1}, x_{n+1} - (x^* + rh)\rangle + \frac{1}{2}\|x_n - x_{n+1}\|^2 \geq 0. \tag{3.13}$$

From (3.13), we have

$$\begin{aligned} r\langle h, x_n - x_{n+1}\rangle &\leq \langle x_n - x_{n+1}, x_{n+1} - x^*\rangle + \frac{1}{2}\|x_n - x_{n+1}\|^2 \\ &= \frac{1}{2}\Big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big) \end{aligned}$$

and then

$$\langle h, x_n - x_{n+1}\rangle \leq \frac{1}{2r}\Big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big). \tag{3.14}$$

Taking sup over $\|h\| \leq 1$, we have

$$\|x_n - x_{n+1}\| \leq \frac{1}{2r}\Big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big) \tag{3.15}$$

Thus for $n > m$, the following inequality holds:

$$\begin{aligned} \|x_n - x_m\| &\leq \sum_{i=m}^{n} \|x_{i+1} - x_i\| \\ &\leq \sum_{i=m}^{n} \frac{1}{2r}\Big(\|x_i - x^*\|^2 - \|x_{i+1} - x^*\|^2\Big) \\ &= \frac{1}{2r}\Big(\|x_m - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big). \end{aligned}$$

Since the sequence $\{\|x_n - x^*\|\}$ has a limit, we have that $\|x_n - x_m\| \to 0$ as $n, m \to \infty$, and thus the sequence $\{x_n\}$ is Cauchy. Since H is complete, and K closed, there exists $p \in K$ such that $x_n \to p$. Since T is continuous, $Tx_n \to Tp$. Thus $\|p - Tp\| = \lim_{n\to} \|x_n - Tx_n\| = 0$ and therefore $Tp = p$. □

4 Conclusion

Most important iteration procedures currently in the literature [12], can be summarised as follows:

(1) $x_{n+1} = Tx_n, \;\; n \geq 0$ 1890 Picard

$\Uparrow \lambda = 1$

(2) $x_{n+1} = \frac{1}{2}(x_n + Tx_n), n \geq 0 \geq 0$ 1955 Krasnoselski

$\Uparrow \lambda = \frac{1}{2}$

(3) $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\; n \geq 0,\; 0 \leq \lambda \leq 1,$ 1957 (Krasnoselski-)Shaeffer

$\Uparrow a_n = \lambda(const.)$

(4) $x_{n+1} = (1 - a_n)x_n + a_n Tx_n, n \geq 0, a_n \in [0,1],$

$$\lim_{n\to\infty} a_n = 0, \sum a_n = \infty \qquad \text{1953 Mann}$$

$$\Uparrow b_n = 0$$

$$(5)\ \ x_{n+1} = (1-a_n)x_n + a_n T[(1-b_n)x_n + b_n T x_n],\ n \geq 0, 0 \leq a_n \leq b_n \leq\ 1,$$

$$\lim_{n\to} b_n = 0,\ \ \sum_{n=0}^{\infty} a_n b_n = \infty \qquad \text{1974 Ishikawa}$$

Our iterative procedure fills the gap between (4) and (5) above in the sense that here $a_n = b_n = \lambda$ simply for some $\lambda \in (0, L^{-2}[\sqrt{1+L^2}-1])$ and the reccurence formula is equally applicable to the class of Lipschitz pseudocontractive mappings to which the Ishikawa procedure is applicable.
In this regard, Corollary 3.3 is an improvement of the result of Ishikawa (Theorem 1.1) in the sense that a similar conclusion is obtained using a simpler recursion formula (3.1).

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Competing Interests

The author declares that no competing interests exist.

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