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Different Levels of Perturbations of the Operators of Hammerstien's Type Operator Equations

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Abstract

We have studied perturbations of Hammerstein's Type Operator Equations in general Banach Spaces. In this paper, two different levels of perturbations have been studied in Hilbert spaces. We prove that these levels satisfy the regularization conditions for Hammerstein type operator equations.

Keywords: Maximal monotone mappings; hammerstein operator equations; regularization; general banach space; hilbert space.

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1 Introduction and Preliminaries

A non-linear integral of the form:

$$u(x) + \int_{\Omega} k(x,y)f(y,u(y))dy = w(x) \tag{1.1}$$

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where Ω is a domain of $\sigma - finite$ measure dy in $\mathcal{R}^N$, the kernel $k : \Omega \times \Omega \mapsto \mathcal{R}$ and the given function $f : \Omega \times \mathcal{R} \mapsto \mathcal{R}$ are both measurable; the unknown function u and the inhomogenous function w lie in a given Banach space X of meausrable real-valued functions on Ω was studied in [1].
Setting the linear operator

$$Bv = \int_\Omega k(., y)v(y)dy \ ,$$

the Nemitskyi or superposition operator

$$Au = f(., u(.)),$$

then the Hammerstein's equation (1.1) is written in operator forms as:

$$u + BAu = w. \tag{1.2}$$

The Nemitskyi operator A is well-defined on a given space X of functions on Ω, and that for each element u of X, $A(u)$ lies in a conjugate space X^* of the space X, the composition BA of the two operators is well-defined and maps X into itself. Given w in the function space X, the integral equation then asks for some u in X such that $(I + BA)(u) = w$. It is noted that if B and A are monotone, then $K := I + BA$ needs not necessarily be monotone. If $X = H$, a Hilbert space, B and A are monotone and if B is compact, [2] proved that a suitably defined Galerkin approximation converges strongly to a solution of :

$$u + BAu = 0. \tag{1.3}$$

Interest in (1.2) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can, as a rule, be transformed into a form (1.2) see [3]. In this paper, we consider X a reflexive Banach space, A, B maximal monotone operators and the equations of study, notably the operator equation of the Hammerstein's type (1.2). The operators A, B will be regularized or perturbed by some small parameter. The regularized equations will finally be examined in Hilbert spaces so as to unify results.The methodology of this paper is to apply regularized techniques to solve operator equations of the Hammerstein's type. This approach was used in [4] where conditions for regularization techniques were discussed.

A set $M \subseteq X \times X^*$ is monotone provided that $\langle f - g, u - v\rangle \geq 0$ for any pair $[u, f], [v, g] \in M$. A monotone set M is maximal if it is not a proper subset of a monotone set in $X \times X^*$. The mapping $T : X \to 2^{X^*}$ is said to be maximal monotone if its graph $G(T)$ is a maximal monotone set of $X \times X^*$. Therefore, T is maximal monotone if and only if $\langle f - g, u - v\rangle \geq 0$ implies $u \in D(T)$ and $f \in Tu$. The element $[u, f] \in X \times X^*$ lies in $G(T)$ if and only if $[f, u] \in X^* \times X$ lies in $G(T^{-1})$. Since the monotonicity is invariant under transposition of domain and the range of a map, T is maximal monotone if and only if T^{-1} has this property.
Let X be an arbitrary Banach space with dual space X^* . The mapping J where $J : X \to 2^{X^*}$ is called a normalized duality mapping on X and is defined by

$$J(u) = \{u^* \in X^* : \langle u^*, u\rangle = ||u||^2 = ||u^*||^2\}. \tag{1.4}$$

Let X be a Reflexive Banach Space with dual X^* , the mappings $A : X \to 2^{X^*}$ and $B : X^* \to 2^X$ are maximal monotone with respective domains $D(A)$ and $D(B)$.
In (1.2) we define a map $C = A^{-1} : X^* \to 2^X$ and for any $\varphi \in Au$, (1.2) may be written as:

$$C\varphi + B\varphi = w \tag{1.5}$$

where C is maximal monotone [3], p 122

Theorem 1.1 . Suppose that X is reflexive, that A and B are maximal monotone operators on X and that

$$D(A) \cap intD(B) \neq \emptyset \tag{1.6}$$

then $A + B$ is maxima.l

Proof. See [5] .

□

Combining (1.5) with the two theorems in [3] p106 - 107, we have the following:

Lemma 1.2 If $A : X \to 2^{X^*}$ and $B : X^* \to 2^X$ maximal monotone mappings, the map $A^{-1} + B$ coercive and the condition (1.6) holds, then (1.2) or (1.5) with any $w \in X$ has at least one solution.

The formulation of a regularized equation of Hammerstein's type requires the following definitions and notations.

Definition 1.3 Let the space $\mathcal{Z}$ be defined by

$$\mathcal{Z} = X \times X^* = \{\varsigma = [u, \varphi] : u \in X, \varphi \in X^*\}. \tag{1.7}$$

With the natural linear operation, + , defined by $\alpha\varsigma_1 + \beta\varsigma_2 = [\alpha u_1 + \beta u_2, \alpha\varphi_1 + \beta\varphi_2]$ for real numbers α and β and $\varsigma_1 = [u_1, \varphi_1], \varsigma_2 = [u_2, \varphi_2]$ are elements of $\mathcal{Z}$.
For any $\varsigma \in \mathcal{Z}$, let $||\varsigma||_{\mathcal{Z}} = \{||u||^2 + ||\varphi||^2\}^{\frac{1}{2}}$ then the space $\mathcal{Z}$ is a Banach space with $\mathcal{Z}^* = X^* \times X$ as its dual. The duality pairing of the spaces $\mathcal{Z}$ and $\mathcal{Z}^*$ is defined by the product

$$\langle \eta^*, \varsigma \rangle = \langle \psi, u \rangle + \langle \varphi, v \rangle \tag{1.8}$$

of the elements $\varsigma = [u, \varphi] \in \mathcal{Z}$ and $\eta^* = [\psi, v] \in \mathcal{Z}^*$.

Lemma 1.4 Let $\{\varsigma_n\}$ be a sequence in $\mathcal{Z}$, where $\varsigma_n = [u_n, \varphi_n]$ and let $\varsigma_0 = [u_0, \varphi_0]$. As $n \to \infty$, the following relations are equiavelent:

$$\varsigma_n \rightharpoonup \varsigma_0, ||\varsigma_n||_{\mathcal{Z}} \to ||\varsigma_0||_{\mathcal{Z}} \tag{1.9}$$

and

$$u_n \rightharpoonup u_0, \varphi_n \rightharpoonup \varphi_0, ||u_n|| \to ||u_0||, ||\varphi_n|| \to ||\varphi_0||. \tag{1.10}$$

Lemma 1.5 If $J : X \to X^*$ and $J^* : X^* \to X$ are the normalized duality mapping on X and X^*, respectively, then the operator $J_{\mathcal{Z}} : \mathcal{Z} \mapsto \mathcal{Z}^*$, defined by

$$J_{\mathcal{Z}}\varsigma = [Ju, J^*\varphi] \quad \forall \varsigma = [u, \varphi] \in \mathcal{Z} \tag{1.11}$$

is a normalized duality mapping on $\mathcal{Z}$; and conversely, every normalized duality mapping on $\mathcal{Z}$ has the form (1.11).

Proof. See [4].

□

The equation (1.2) is written as the system:

$$Au - \varphi = 0, u + B\varphi = w, \varphi \in Au$$

which is equivalent to the operator equation:

$$T\varsigma = h \tag{1.12}$$

where $T : \mathcal{Z} \mapsto 2^{\mathcal{Z}^*}$ such that

$$T\varsigma = [Au - \varphi, u + B\varphi] \in \mathcal{Z}^*, \varsigma = [u, \varphi] \in \mathcal{Z}, h = [0_{X^*}, \omega] \in \mathcal{Z}^*$$

Lemma 1.6 The operator T is monotone.

Proof. Let $\varsigma_1 = [u_1, \varphi_1], \varsigma_2 = [u_2, \varphi_2] \in \mathcal{Z}$, the equality

$$\langle T\varsigma_1 - T\varsigma_2, \varsigma_1 - \varsigma_2\rangle = \langle Au_1 - Au_2, u_1 - u_2\rangle + \langle \varphi_1 - \varphi_2, B\varphi_1 - B\varphi_2\rangle \tag{1.13}$$

holds . Additionally, the equality is valid if $u_i \in D(A), \varphi_i \in D(B)$ for all $i = 1, 2$. At that, the inclusion $\varphi_i \in Au_i$ should hold, that is, $u_i \in A^{-1}\varphi_i, i = 1, 2$.
Therefore (1.13) may be written as
$\langle T\varsigma_1 - T\varsigma_2, \varsigma_1 - \varsigma_2\rangle = \langle \varphi_1 - \varphi_2, A^{-1}\varphi_1 - A^{-1}\varphi_2\rangle + \langle \varphi_1 - \varphi_2, B\varphi_1 - B\varphi_2\rangle$
where $\varphi_i \in D(B), \varphi_i \in D(A^{-1}), i = 1, 2.$
With this condition it follows that if (1.6) is satisfied then the operator T is maximal monotone on its domain $D(T)$.

□

Let the operators A, B be maximal monotone. Let $D(A) = X$ and condition (1.6) be satisfied. Let N be a non-empty closed solution set of (1.2). Then we have the following result:

Lemma 1.7 The set N is convex.

Proof. See [4].

□

Next we consider a regularized Hammerstein type operator equation in general Banach space

$$u + (B + \alpha J^*)(A + \alpha J)u = w^\delta \tag{1.14}$$

where $\alpha > 0, \delta > 0$, w^δ is $\delta-$approximation of w, where $||w - w^\delta|| \leq \delta$

Lemma 1.8 The equation (1.14) is uniquely solvable for every element $w^\delta \in X$.

Proof. Let $B^\alpha = B + \alpha J^*$, $A^\alpha = A + \alpha J^*$.
We introduce an operator
$T^\alpha = [A^\alpha u - \phi, u + B^\phi] = T\varsigma + \alpha J_{\mathcal{Z}}\varsigma, \varsigma = [u, \phi]$, $J_{\mathcal{Z}}\varsigma = [Ju, J^*\phi]$
The solvability of (1.14) is equivalent to the solvability of equation

$$T^\alpha \varsigma_\alpha^\delta = h^\delta, \qquad h^\delta = [0^*_{\mathcal{X}}, \omega^\delta] \tag{1.15}$$

where $T^\alpha = T + \alpha J_{\mathcal{Z}}$ is a maximal monotone. The conclusion of the Lemma follows from (see [6]).

□

Theorem 1.9 Let a solution set N of equation (1.2) be non empty and closed, $A : X \to X^*$ be a maximal monotone locally bounded mapping with $D(A) = X$. Assume that $B : X^* \mapsto 2^X$ is also maximal monotone mapping. Let w^δ be a $\delta-$ approxiamtion of w, such that $||w - w^\delta|| \leq \delta$. If $\frac{\delta}{\alpha} \to 0$, and also $\alpha \to 0$, then the sequence $\{u_\alpha^\delta\}$ of solutions of the regularized equation (1.14) converges strongly in X to the solution $u^* \in N$ which is defined

$$||u^*||^2 + ||Au^*||^2 = min\{||u||^2 + ||Au||^2 : u \in N\}.$$

Proof. See [4].

□

2 New Results

We now discuss our results in Hilbert spaces by looking at different levels of small perturbations of the parameters of the Hammerstein's type operator equation.

2.1 Case 1

The following operators: $A, B, B^{h_1}, A^{h_2} : H \mapsto 2^H$ are maximal monotone mappings.
Let the error estimates of the operators be $||B^{h_1}u - Bu||_H \leq h_1 g(||u||); ||A^{h_2}u - Au||_H \leq h_2 g(||u||); ||w^\delta - w||_H \leq \delta$; where $\delta > 0$, $h_1, h_2 > 0$ and g is a functional.
Let the regularized Hammerstein type equation be given by:

$$u + (B^{h_1} + \alpha I_H)(A^{h_2} + \alpha I_H)u = w^\delta. \tag{2.1}$$

Then equation (1.2) is put in the form:

$$T\varsigma = h \tag{2.2}$$

where $T\varsigma = [Au - \phi, u + B\phi], h = [0, w], \zeta = [u, \phi]$.
The equation (2.1) is also represented by:

$$T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{h_1h_2} = h^\delta_{h_1h_2} \tag{2.3}$$

where

$$T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{h_1h_2} = h^\delta_{h_1h_2} = [(A^{h_2} + \alpha I_H)u^{\alpha\delta}_h - \phi^{\alpha\delta}_{h_1h_2}, u^{\alpha\delta}_{h_1h_2} + (B^{h_1} + \alpha I_H)\phi^{\alpha\delta}_{h_1h_2}],$$

$$h^\delta_{h_1h_2} = [0, w^\delta]$$

and

$$\varsigma^{\alpha\delta}_{h_1h_2} = [u^{\alpha\delta}_{h_1h_2}, \phi^{\alpha\delta}_{h_1h_2}].$$

Furthermore

$$T^{\alpha\delta}_{h_1h_2} = T^\delta_{h_1h_2} + \alpha I_{\mathcal{Z}}, \mathcal{Z} = H \times H \tag{2.4}$$

where

$$T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{h_1h_2} = [A^{h_2}u^{\alpha\delta}_{h_1h_2} - \phi^{\alpha\delta}_{h_1h_2}, u^{\alpha\delta}_{h_1h_2} + B^{h_1}\phi^{\alpha\delta}_{h_1h_2}]$$

and

$$I_{\mathcal{Z}} = [I_H u^{\alpha\delta}_h, I_H \phi^{\alpha\delta}_h].$$

Let $\varsigma^{\alpha\delta}_{(h_1h_2)_1}, \varsigma^{\alpha\delta}_{(h_1h_2)_2} \in \mathcal{Z}$. Then clearly

$$0 \leq \langle T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{(h_1h_2)_1} - T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{(h_1h_2)_2}, \varsigma^{\alpha\delta}_{(h_1h_2)_1} - \varsigma^{\alpha\delta}_{(h_1h_2)_2}\rangle$$

that is $T^{\alpha\delta}_{h_1h_2}$ is monotone. By Lemma 3.2 in [4], (2.3) is uniquely solvable for $T^{\alpha\delta}_{h_1h_2}$ which is maximal monotone.
We verify the requirements inherent in Theorem 1.9 . We subtract (1.12) from (2.3) and multiply throughout by $\varsigma^{\alpha\delta}_{h_1h_2} - \varsigma$ to obtain

$$\langle T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{h_1h_2} - T\varsigma, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle = \langle h^{\delta}_{h_1h_2} - h, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle$$

or

$$\langle T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{h_1h_2} - T\varsigma, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle + \alpha\langle I_{\mathcal{Z}}\varsigma^{\alpha\delta}_{h_1h_2}, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle = \langle h^{\delta}_{h_1h_2} - h, \varsigma^{\alpha\delta}_{h} - \varsigma\rangle$$

implying that

$$\alpha\langle\varsigma^{\alpha\delta}_{h_1h_2}, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle = \langle h^{\delta}_{h_1h_2} - h, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle - \langle T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{h_1h_2} - T\varsigma, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle. \tag{2.5}$$

However

$$-\langle T^{\alpha\delta}_{h_1h_2}\varsigma^{\alpha\delta}_{h_1h_2} - T\varsigma, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle \leq hg(||u||)||u^{\alpha\delta}_{h_1h_2} - u|| + h(g(||\phi||)||\phi^{\alpha\delta}_{h_1h_2} - \phi||).$$

Also

$$\langle h^{\delta}_{h_1h_2} - h, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle \leq \delta||\varsigma^{\alpha\delta}_{h_1h_2} - \varsigma||.$$

Therefore from (2.5) we have:

$$\langle\varsigma^{\alpha\delta}_{h_1h_2}, \varsigma^{\alpha\delta}_{h_1h_2} - \varsigma\rangle \leq \tfrac{\delta}{\alpha}||\varsigma^{\alpha\delta}_{h_1h_2} - \varsigma||_{\mathcal{Z}} + \tfrac{h_1h_2}{\alpha}[g(||u||)||u^{\alpha\delta}_{h_1h_2} - u|| + g(||\phi||)||\phi^{\alpha\delta}_{h_1h_2} - \phi||]$$

or equivalently

$$||\varsigma^{\alpha\delta}_{h_1h_2}||^2 - ||\varsigma^{\alpha\delta}_{h_1h_2}||||\varsigma|| \leq \tfrac{\delta}{\alpha}||\varsigma^{\alpha\delta}_{h_1h_2}||_{\mathcal{Z}} + ||\varsigma||_{\mathcal{Z}} + \tfrac{h_1h_2}{\alpha}[g(||u||)||u^{\alpha\delta}_{h_1h_2} - u|| + g(||\phi||)||\phi^{\alpha\delta}_{h_1h_2} - \phi||],$$

which is quadratic expression in $||\varsigma^{\alpha\delta}_{h_1h_2}||$. An argument similar to the proof of Theorem (1.9) shows that $||\varsigma^{\alpha\delta}_{h_1h_2}||$ is bounded and therefore there exists a subset $\{\varsigma^{\beta\delta}_{h_1h_2}\}$ which converges as $\beta \to 0$ to some element $\bar{\varsigma} \in \mathcal{Z}$.
Next we show that $\bar{\varsigma}$ is a solution of (1.12). Since T is monotone, we have
$0 \leq \langle T\varsigma - T\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle = \langle T\varsigma - T^{\delta}_{h_1h_2}\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + \langle T^{\delta}_{h_1h_2}\varsigma^{\beta\delta}_{h_1h_2} - T\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle$

$= \langle T\varsigma - h, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + \langle h - h^{\delta}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + \langle h^{\delta} - T^{\delta}_{h_1h_2}\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + \langle T^{\delta}_{h_1h_2}\varsigma^{\beta\delta}_{h_1h_2} - T\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle$

$\leq \langle T\varsigma - h, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + \delta||\varsigma - \varsigma^{\beta\delta}_{h_1h_2}|| + \beta\langle I_{\mathcal{Z}}\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + \langle T^{\delta}_{h_1h_2}\varsigma^{\beta\delta}_{h_1h_2} - T\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle$
However,

$$\langle T^{\delta}_{h_1h_2}\varsigma^{\beta\delta}_{h_1h_2} - T\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle \leq h[g(||u^{\beta\delta}_{h}||)||u - u^{\beta\delta}_{h_1h_2}|| + g(||\phi^{\beta\delta}_{h_1h_2}||)||\phi - \phi^{\beta\delta}_{h_1h_2}||].$$

Substituting we have
$0 \leq \langle T\varsigma - h, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + \delta||\varsigma - \varsigma^{\beta\delta}_{h_1h_2}|| + \beta\langle I_{\mathcal{Z}}\varsigma^{\beta\delta}_{h_1h_2}, \varsigma - \varsigma^{\beta\delta}_{h_1h_2}\rangle + h[g(||u^{\beta\delta}_{h_1h_2}||)||u - u^{\beta\delta}_{h_1h_2}|| + g(||\phi^{\beta\delta}_{h_1h_2}||)||\phi - \phi^{\beta\delta}_{h_1h_2}||]$.

As $\beta, \delta, h_1, h_2 \to 0$, we have

$$0 \leq \langle T\varsigma - h, \varsigma - \bar{\varsigma}\rangle\ .$$

Hence by the monotonicity of T, we have

$$T\bar{\varsigma} = h.$$

Finally from (2.5),
$\langle \varsigma_{h_1h_2}^{\alpha\delta}, \varsigma_{h_1h_2}^{\alpha\delta} - \varsigma \rangle \leq \frac{\delta}{\alpha}||\varsigma_{h_1h_2}^{\alpha\delta} - \varsigma||_{\mathcal{Z}} + \frac{h_1h_2}{\alpha}[g(||u||)||u_{h_1h_2}^{\alpha\delta} - u|| + g(||\phi||)||\phi_{h_1h_2}^{\alpha\delta} - \phi||]$.

As $\frac{\delta}{\alpha}, \frac{h_1}{\alpha}, \frac{h_2}{\alpha} \to 0$, we have

$$\langle \varsigma_{h_1h_2}^{\alpha\delta}, \varsigma_{h_1h_2}^{\alpha\delta} - \varsigma \rangle \leq 0$$

implying that

$$||\varsigma_h^{\alpha\delta}|| \leq ||\varsigma||.$$

Hence

$$||\varsigma_{h_1h_2}^{\alpha\delta}|| \leq min_{\varsigma \in N}||\varsigma||.$$

Therefore Theorem 1.9 is satisfied.

2.2 Case 2

Let $\alpha, \beta > 0$ be small real numbers. We study the regularized Hammerstein type operator equation:

$$u + (B + \beta I_{\mathcal{H}})(A + \alpha I_{\mathcal{H}})u = \omega \tag{2.6}$$

Then equation (2.2) is put in the form

$$T^{\alpha\beta}\varsigma_{\alpha\beta} = h \tag{2.7}$$

where

$$T^{\alpha\beta}\varsigma_{\alpha\beta} = [A^{\alpha}u_{\alpha\beta} - \phi_{\alpha\beta}, u_{\alpha\beta} + B^{\beta}\phi_{\alpha\beta}], h = [0_{\langle}, \omega],$$

$$\varsigma_{\alpha\beta} = [u_{\alpha\beta}, \phi_{\alpha\beta}] \in \mathcal{Z}$$

and

$$A^{\alpha} = A + \alpha I_{\mathcal{H}}, B^{\beta} = B + \beta I_{\mathcal{H}}.$$

Therefore

$$T^{\alpha\beta}\varsigma_{\alpha\beta} = [A^{\alpha}u_{\alpha\beta} - \phi_{\alpha\beta}, u_{\alpha\beta} + B^{\beta}\phi_{\alpha\beta}] + [\alpha I_{\mathcal{H}}u_{\alpha\beta}, \beta I_{\mathcal{H}}\phi_{\alpha\beta}]$$

$$= T\varsigma_{\alpha\beta} + [\alpha I_{\mathcal{H}}u_{\alpha\beta}, \beta I_{\mathcal{H}}\phi_{\alpha\beta}] := T\varsigma_{\alpha\beta} + I_{\alpha\beta}\varsigma_{\alpha\beta}$$

Since T is monotone, for any ς , $\varsigma_{\alpha\beta} \in \mathcal{Z}$, we have

$$0 \leqslant \langle T\varsigma - T\varsigma_{\alpha\beta}, \varsigma - \varsigma_{\alpha\beta} \rangle.$$

$$= \langle T\varsigma - h + I_{\alpha\beta}\varsigma_{\alpha\beta}, \varsigma - \varsigma_{\alpha\beta} \rangle$$

$$= \langle T\varsigma - h, \varsigma - \varsigma_{\alpha\beta}\rangle + \langle I_{\alpha\beta}\varsigma_{\alpha\beta}, \varsigma - \varsigma_{\alpha\beta}\rangle.$$

As $\alpha, \beta \to 0$; $\langle I_{\alpha\beta}\varsigma_{\alpha\beta}, \varsigma - \varsigma_{\alpha\beta}\rangle \to 0$ and therefore

$$0 \leqslant \langle T\varsigma - h, \varsigma - \varsigma_{\alpha\beta}\rangle$$

implying that

$$T\varsigma_{\alpha\beta} = h$$

that is, $\varsigma_{\alpha\beta}$ is a solution of (2.2). Next as we subtract (2.2) from (2.7) , we have

$$\langle T_{\alpha\beta}\varsigma_{\alpha\beta} - T\varsigma, \varsigma_{\alpha\beta} - \varsigma\rangle = 0.$$

Simplifying this last equation, we have:

$$\alpha||u_{\alpha\beta}||^2 + \beta||\phi_{\alpha\beta}||^2 \leqslant \alpha\langle u_{\alpha\beta}, u\rangle + \beta\langle\phi_{\alpha\beta}, \phi\rangle. \tag{2.8}$$

However with

$$\langle u_{\alpha\beta}, u\rangle \leqslant \tfrac{1}{2}[||u_{\alpha\beta}||^2 + ||u||^2]$$

$$\langle \phi_{\alpha\beta}, \phi\rangle \leqslant \tfrac{1}{2}[||\phi_{\alpha\beta}||^2 + ||\phi||^2]$$

(2.8) becomes

$$\alpha||u_{\alpha\beta}||^2 + \beta||\phi_{\alpha\beta}||^2 \leqslant \alpha||u||^2 + \beta||\phi||^2. \tag{2.9}$$

Following a similar argument from **Case I**, it is clearly seen that

$$||\varsigma_{\alpha\beta}|| \leq min_{\varsigma\in N}||\varsigma||.$$

3 Conclusions

In this work, we have studied two levels of perturbations in Hilbert spaceof Hammerstein type operator equations. In Subsection 2.1 the perturbations on the operators A and B were chosen at different levels while in Subsection 2.2 the perturbations on the normalized duality mapswhich are adjoined to the operators A and B. These regularized equations can be used to find approximate solutions of the non-linear Hammerstein integral equations of the form (1.1).

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Competing Interests

Authors have declared that no competing interests exist.

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