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Augmented Lagrangian Method for One Dimensional Optimal Control Problems Governed by Delay Differential Equation

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Abstract

In this research, numerical solutions of continuous optimal control problems governed by linear damping evolution with delay and real coefficients are presented. The necessary conditions obtained from the knowledge of calculus of variation for optimal control problem constrained by delay differential equation is a linear two-point boundary value problem involving both delay and advance terms. Clearly, this coupling that exists between the state variable and the control variable is not amenable to analytical solution hence a direct numerical approach is adopted. We propose an augmented discretized continuous algorithm via quadratic programming, which is capable of handling optimal control problems constrained by delay differential equations. The discretization of the problem using trapezoidal rule (a one step second order numerical scheme) and Crank-Nicholson with quadratic formulation amenable to quadratic programming technique for solution of the optimal control problems are considered. A control operator (penalized matrix), through the augmented Lagrangian method, is constructed. Important properties of the operator as regards sequential quadratic programming techniques for determining the optimal point are shown.

Keywords: Trapezoidal rule, Cranck-Nicholson, Augmented Lagrangian, Conjugate Gradient Method.

1 Introduction

Optimization is the act of obtaining the best result under given circumstances. In design, construction and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit. Since the effort required

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or the benefit desired in any practical situation can be expressed as a function of certain decision variables, optimization can be defined as the process of finding the conditions that give the maximum or minimum value of a function. It can be taken to mean minimization since the maximum of a function can be found by seeking the minimum of the negative of the same function.

Jamshidi et al. [1] considered the near optimum solution of the following class of linear systems with input-time delay;

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Cu(t-T) \\ u(t) &= \alpha(t), T - t_0 \le t \le T \end{aligned} \tag{2.10}$$

Where $x \in R^1, u \in R^1$ are the state and control vectors, A, B and C are constant matrices of appropriate dimensions, T is the final time, $\alpha(t)$ is the control initial function. They obtained control vector which minimizes a quadratic cost functional;

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x' Qx + u' Ru)du \ , \tag{2.11}$$

where t_0 is the initial time, t_f is the final time, Q *and* R are scalars through the introduction of small parameter ϵ and Maclaurin series expansion. The control has an exact feedback portion and a truncated series open loop gain. For all orders of approximations, only one Riccati equation must be solved and the new approximation needs only the previous history as in some near optimization techniques. Their method was attractive computationally and can be easily extended to nonlinear and time-varying systems.

The function space algorithm constructed by Di-Pillo [2] suffers major set-back in terms of implementation and convergence. This was due to approximation adopted in computing. The difficulty encountered in function space algorithm prompted Bock et al. [3] to adopt the control parameterization techniques in the numerical solution of optimal control problems.

Ibiejugba et al. [4] proposed a Control Operator and some of its Application", discovered the numerical set-back in function space algorithm which can be circumvented in order to reduce its high level of sophistication and make the algorithm accessible to both specialists and non-specialists in control theory. They constructed a control operator (A) which rendered conjugate gradient algorithm amenable to solution of continuous optimal control problems and used their explicit knowledge of the operator to devise an extended conjugate gradient method (ECGM).

Jaechong [5] considered the linear delay differential equation characterized by a quadratic cost functional, where he introduced a new linear operator in such away that the state equation subject to a starting function can be viewed as an inhomogeneous boundary valued problem, and derived the adjoint operator of the new operator, and then define the formal adjoint operator which play an important role in the characterization of the optimal control. Although the method avoids the usual semi-group theory treatment to the problems but only gives the necessary theory for such problems.

Cai et al. [6] developed an optimal control for linear systems with time delay. In the proposed control method, the differential equation with time delay of the system dynamics was first written into a form without any time delay through a particular transformation. Then, the optimal controller was designed using the classical optimal control theory. A numerical algorithm for control implementation was presented. Since the obtained expression of the optimal controller contains an integral term that is not convenient for on-line calculation, the time delay was considered at the very beginning of the control design and no approximation and estimation were made in the control system. Thus the system performance and stability were prone to be guaranteed. Instability in responses might occur only if a system with time delay is controlled by the optimal controller that was designed with no consideration of time delay. They demonstrated the effectiveness of proposed optimal controller by simulation studies.

Smith [7] considered a black box solver using evolutionary algorithm for optimal control problem governed by delay differential equation and the result was better compared to existing algorithms, though it took a longer computational time

Olotu et al. [8] proposed an extended discretized scheme to examine the convergence profile of a quadratic control problem constrained by evolution equation with real coefficients. With an unconstrained formulation of the problem via the penalty-multiplier method, the discretization of the time interval and differential constraint is carried out. An operator, to circumvent the cumbersome calculation inherent in some earlier scheme such as function space algorithm, is established and proved.

Olotu et al. [9], Bock Hans G and Karl JP [3]: A multiple Shooting Algorithm for Direct Solution of Optimal Control Problems. Proceedings of the 9th IFAC World Congress, Budapest developed a discretized algorithm via quadratic programming techniques. In the developed algorithm, the optimal control problem is discretized and through the augmented Lagrangian method, a penalized matrix is constructed to reduce the problem of ill-conditioning. The optimal control problem then becomes large sparse quadratic programming problem ameanable to conjugate gradient method.

In this work, the discretized algorithm is extended to optimal control problems governed by delay differential equations. Where both the objective function and the constraint with delay term are discretized and a control operator (penalized matrix) is formulated which render the problem ameanable to conjugate gradient method.

2. Method of Solution

Consider optimal control problem of the form,

$$\min J(x,u) = \int_0^T (px^2(t) + qu^2(t))dt \tag{2.1}$$

$$\text{Subject to} \begin{cases} \dot{x}(t) = ax(t) + bx(t-r) + cu(t), t \in [0,T] \\ x(t) = h(t), \ \ t \in [-r,0] \end{cases} \tag{2.2}$$

In order to make (2.1) and (2.2) amenable to conjugate gradient method, we shall replace the constrained problem by appropriate discretised optimal control problem.

Breaking the interval $[0,T]$ into n equal intervals with knots $t_0 < t_1 < t_2 < ... < t_n$ and say

$$\Delta t_i = 0.1 \text{. With } h = \frac{T-0}{n} \tag{2.3}$$

Discretise (2.1) using trapezoidal we have,

$$\int_0^T (px^2(t) + qu^2(t))dt = \frac{h}{2}\left[\sum_{k=0}^{T}\left[p(f(x_k) + f(x_{k-1}) + q(f(u_k) + f(u_{k-1})\right]\right]$$

$$\int_0^T (px^2(t) + qu^2(t))dt = \frac{h}{2}\left[\sum_{K=0}^{T}\left[p\left(f(x_k) + f(x_{k-1})\right) + q(f(u_k) + f(u_{k-1}))\right]\right] \tag{2.4}$$

Since $x(t_0) = h(t)$ from equation (2.2)

Generating matrix for the coefficients of x'^s *and* u'^s we have,

$$(x_1 \quad x_2 \quad \cdots \quad x_T \quad u_0 \quad u_1 \quad \cdots \quad u_T)\begin{bmatrix} ph & & & & & & & \\ & ph & & & & & & \\ & & \ddots & & & & & \\ & & & p\frac{h}{2} & & & & \\ & & & & q\frac{h}{2} & & & \\ & & & & & qh & & \\ & & & & & & \ddots & \\ & & & & & & & q\frac{h}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \\ u_0 \\ u_1 \\ \vdots \\ u_T \end{bmatrix} + A_0 \tag{2.5}$$

Where

$$A_0 = \frac{h}{2}ph^2(t) \tag{2.6}$$

This can be re-written in quadratic form as,

$$Z^T MZ + A_0 \tag{2.7}$$

Where Z is a column vector of dimension $1\times(2n+1)$, M is a square matrix of dimension

$$(2n+1)\times(2n+1)$$

Discretising the constraint $\dot{x}(t) = ax(t) + bx(t-r) + cu(t)$, totally using crank-Nicholson and assume that there exist x_k such that $x(t_k - r) = x_{k-r}$. This is true for rational values of r and we ensure that our point of discretization falls exactly on r to avoid off grid points.
Hence,

$$\dot{x} = \frac{x_{k+1} - x_k}{h} = \frac{1}{2}\{ f(x_{k+1}, u_{k+1}) + f(u_k, x_k)\} + o(h)^2 \quad (2.8)$$

$$x_{k+1} = dx_k + e(x_{k+1-r} + x_{k-r}) + f(u_{k+1} + u_k) \quad (2.9)$$

Where $d = \frac{(2+ah)}{(2-ah)}$

$$e = \frac{bh}{(2-ah)}$$

$$f = \frac{ch}{(2-ah)}$$

Now, the resulting discretised optimal control problem is,

$$\min I = \sum_{k=1}^{T} px_k^2 + qu_k^2 \quad (2.10)$$

Subject to: $$x_{k+1} = dx_k + e(x_{k+1-r} + x_{k-r}) + f(u_{k+1} + u_k) \quad (2.11)$$

Considering (2.11), there exist m such that $m = \frac{r}{h}$ $where\ h = \Delta t_k$

For $t \in [-r, 0]$, where $x(t-r) = x_{k-r}$ is known to be constant we have,

When $k = 0$,

$$x_1 = dx_0 + ex_{1-r} + ex_{-r} + fu_1 + fu_0$$

$$x_1 - ex_{1-r} - ex_{-r} - fu_1 - fu_0 = dx_0 \quad (2.12\,i)$$

When $k = 1$,

$$x_2 = dx_1 + ex_{2-r} + ex_{1-r} + fu_2 + fu_1$$

$$x_2 - dx_1 - ex_{2-r} - ex_{1-r} - fu_2 - fu_1 = 0 \quad (2.12\,ii)$$

When $k = 2$,

$$x_3 = dx_2 + ex_{3-r} + ex_{2-r} + fu_3 + fu_2$$

$$x_3 - dx_2 - ex_{3-r} - ex_{2-r} - fu_3 - fu_2 = 0 \qquad (2.12\,iii)$$

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When $k = m-1$,

$$x_m = dx_{m-1} + ex_{m-1-r} + ex_{m-r} + fu_m + fu_{m-1}$$

$$x_m - dx_{m-1} - ex_{m-1-r} - ex_{m-r} - fu_m - fu_{m-1} = 0 \qquad (2.12\,v)$$

Similarly, for $t \in [0,T]$ where $x(t-r) = x_{k-r}$ is unknown, we have,

For $k = m$,

$$x_{m+1} = dx_m + ex_{m+1-r} + ex_{m-r} + fu_{m+1} + fu_m$$

$$x_{m+1} - dx_m - ex_{m+1-r} - ex_{m-r} - fu_{m+1} - fu_m = 0 \qquad (2.12\,vi)$$

When $k = m+1$,

$$x_{m+2} = dx_{m+1} + ex_{m+2-r} + ex_{m+1-r} + fu_{m+2} + fu_{m+1}$$

$$x_{m+2} - dx_{m+1} - ex_{m+2-r} - ex_{m+1-r} - fu_{m+2} - fu_{m+1} = 0 \qquad (2.12\,vii)$$

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When $k = n-1$,

$$x_n = dx_{n-1} + ex_{n-r} + ex_{n-1-r} + fu_n + fu_{n-1}$$

$$x_n - dx_{n-1} - ex_{n-r} - ex_{n-1-r} - fu_n - fu_{n-1} = 0 \qquad (2.12\,viii)$$

Generating an augmented matrix from the system of equation $(2.12i)$ – $(2.12viii)$ we have,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -f & -f & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -d & 1 & & & & & & & 0 & -f & -f & & & & & & 0 \\ 0 & -d & 1 & & & & & & 0 & 0 & -f & -f & & & & & 0 \\ 0 & & -d & 1 & & & & & 0 & 0 & 0 & -f & -f & & & & 0 \\ -e & & & -d & 1 & & & & 0 & 0 & 0 & & -f & -f & & & 0 \\ 0 & -e & -e & 0 & -d & 1 & & & 0 & 0 & 0 & & & -f & -f & & \vdots \\ \vdots & & -e & -e & 0 & -d & 1 & & \vdots & \vdots & 0 & & & & -f & -f & 0 \\ 0 & 0 & 0 & -e & -e & 0 & -d & 1 & 0 & 0 & 0 & 0 & 0 & & & -f & -f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \\ u_0 \\ u_1 \\ \vdots \\ u_m \\ u_{m+1} \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} dx_0 + ex_{-r} + ex_{1-r} \\ ex_{2-r} + ex_{1-r} \\ \vdots \\ ex_{m-1-r} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (2.13)$$

This can be written as

$$JZ = W \qquad (2.14)$$

Where J is a block matrix of dimension $n\times(2n+1)$ with representation $J=\left(C\vdots E\right)$ where $\left[C\right]$ is an $n\times n$ sparse matrix with principal diagonal elements $\left[C\right]_{ii}$ =1, and lower diagonal elements $-d$ for every i,j such that $i=j+1$. $\left[E\right]$ is $n\times(n+1)$ bidiagonal matrix with $\left[E\right]_{ii}=-f$ and upper diagonal elements $-f$ for every i,j such that $j=i+1$. The column vector $\left[W\right]$ is of order $n\times 1$ with entries given by $\left[W\right]_{11}=cx_0+ex_{-r}+ex_{1-r}$ and $\left[W\right]_{i1}=ex_{i-1-r}$ $i=2,3,...,m$ and $\left[W\right]_{i1}=0$, $i=m+1,m+2,...,n$. $\left[Z\right]$ is also a column vector of dimension $\left(2n+1\right)\times 1$.

Where J is of dimension $n\times(2n+1)$, Z is of dimension $(2n+1)\times 1$ and W is of dimension $n\times 1$

Hence by parameter optimization, the discretised optimal problem becomes a large sparse quadratics programming problem written as,

$$\min I(z)=Z^TMZ+A_0 \tag{2.15}$$

subject to

$$DZ=W \tag{2.16}$$

The unconstrained minimization problem by augmented Lagrangian function is,

$$\min L_p(z)=Z^TMZ+A_0+\lambda^T\left|DZ-W\right|+\frac{1}{\mu}\left\|DZ-W\right\|^2 \tag{2.17}$$

On expansion we have,

$$\min L_p(z)=Z^T(M+\frac{1}{\mu}D^TD)Z+(\lambda^TD-\frac{2}{\mu}W^TD)Z+(A_0-\lambda^TW+\frac{1}{\mu}W^TW) \tag{2.18}$$

$$\min L_p(z)=Z^TA_PZ+BZ+C \tag{2.19}$$

Where $A_p=(M+\frac{1}{\mu}D^TD)$, $B=(\lambda^TD-\frac{2}{\mu}W^TD)$ and $C=(A_0-\lambda^TW+\frac{1}{\mu}W^TW)$

Equation (2.19) is the quadratic programming problem which is solvable using conjugate gradient method.

***Theorem* 3.3:** Considering the formulated quadratic function in equation (2.19), where $A_\rho \in \mathbb{R}^{(2n+1)\times(2n+1)}$, the penalized matrix $A_\rho = \left[M + \frac{1}{\mu}D^T D\right]$ is said to be positive definite if it obeys these properties:

(1) If A_ρ is real

(2) If A_ρ symmetric

(3) If the principal minors of A_ρ are positive.

Proof:

(1) Since for every $a_{i,j} \in A_\rho,\ a_{i,j} \in \mathbb{R}$ It implies thatA_ρ is real

(2) Matrix A_ρ is said to be symmetric if $(A_\rho)^T = \left[M + \frac{1}{\mu}D^T D\right]^T = A_\rho = \left[M + \frac{1}{\mu}D^T D\right]$. Since A is a positive symmetric diagonal matrix. Then

$$(A_\rho)^T = \left[M + \frac{1}{\mu}D^T D\right]^T = \left[(\frac{1}{\mu}D^T D)^T + M^T\right]$$

$$= \left[(\frac{1}{\mu}D^T D)^T + M^T\right] = \left[M^T + \frac{1}{\mu}(D^T D)^T\right]$$

$(commutativitylaw)$

$$(A_\rho)^T = \left[M + \frac{1}{\mu}D^T D\right]^T = \left[M^T + \frac{1}{\mu}(D^T D)^T\right] = \left[M + \frac{1}{\mu}D^T D\right]$$

Hence, $(A_\rho)^T = A_\rho$

(3) Let M_i represents the leading principal minor ofA_ρ, then $\forall i = 2,3,...,$

$$|M_i| = \left|\begin{bmatrix} M_{i-1} & \alpha_i \\ \alpha_i^T & m_{ii} \end{bmatrix}\right| > 0$$

Where $$\alpha_i = \begin{bmatrix} m_{1,i} \\ \vdots \\ m_{i-1,i} \end{bmatrix}$$

Where $m_{ii}^{'s}$ are the last entries of the principal minors (M_i) provided $i \geq 2$

Since

$$A_\rho = \begin{bmatrix} ph+1+C^2 & -C & 0 & 0 & 0 & -D & CD & 0 & 0 & \cdots & 0 \\ -C & ph+1+C^2 & -C & 0 & 0 & 0 & -D & CD & & & 0 \\ 0 & -C & ph+1+C^2 & \ddots & \vdots & 0 & 0 & \ddots & \ddots & & 0 \\ \vdots & 0 & \ddots & \ddots & -C & \vdots & \vdots & \cdots & D & CD & \vdots \\ 0 & 0 & \cdots & -C & p\frac{h}{2}+1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -D & 0 & 0 & \cdots & 0 & D^2+q\frac{h}{2} & 0 & \cdots & 0 & 0 & 0 \\ CD & -D & 0 & \cdots & 0 & 0 & D^2+q\frac{h}{2} & 0 & & \cdots & 0 \\ 0 & CD & \ddots & & & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \ddots & \ddots & -D & 0 & \cdots & \cdots & 0 & \ddots & \ddots & \\ \vdots & 0 & 0 & CD & \vdots & \vdots & \vdots & 0 & 0 & D^2+q\frac{h}{2} & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & & \cdots & 0 & q\frac{h}{2} \end{bmatrix},$$

$A_\rho \in \mathbb{R}^{(2n+1)\times(2n+1)}$

Since all the entries are real, the matrix is said to be real. The matrix is also symmetric since $(A_\rho)^T = A_\rho$.

The submatrices are:

When $i = 1$, we have,

$$|M_1| = |ph+1+C^2| > 0, \quad \forall p > 0$$

When $i = 2$, we have,

$$|M_2| = \left| \begin{bmatrix} ph+1+C^2 & -C \\ -C & ph+1+C^2 \end{bmatrix} \right|$$

$$|M_2| = (ph)^2 + 2C^2ph + 2ph + C^2 + C^4 + 1$$

$$|M_2| = (ph)^2 + 2C^2ph + 2ph + C^2 + C^4 > 0, \quad \forall p > 0$$

Hence, M_2 is positive.

When $i = 3$, we have,

$$|M_3| = \left| \begin{bmatrix} ph+C^2+1 & -C & 0 \\ -C & ph+C^2+1 & -C \\ 0 & -C & ph+C^2+1 \end{bmatrix} \right|$$

$$= ph+C^2+1 \begin{bmatrix} ph+1+C^2 & -C \\ -C & ph+1+C^2 \end{bmatrix} + C \begin{bmatrix} -C & -C \\ 0 & ph+1+C^2 \end{bmatrix}$$

$$= ph+1+C^2\left[(ph+1+C^2)(ph+1+C^2)-C^2\right]-C^2(ph+1+C^2)$$
$$= ph+1+C^2\left[(ph)^2+ph+C^2ph+ph+1+C^2+C^2ph+C^4\right]-C^2ph-C^4-C^2$$
$$=(ph)^3+(ph)^2+3C^2(ph)^2+2(ph)^2+3ph+4C^2(ph)+3C^4(ph)+C^6+C^4+C^2+1$$

Since $p,h>0$,

$$=(ph)^3+(3+3C^2)((ph)^2)+(4C^2+3C^4+3)ph+C^6+C^4+C^2+1>0$$

Hence, $|M_3|>0$.

Hence by mathematical induction, if it's true for values of $i=1,2,3,\cdots$, then, assume it's true for $i=k$, then we shall proof that it is true for $i=k+1$ and we have,

$$|M_{k+1}|=\left|\begin{bmatrix} M_k & \alpha_{k+1} \\ \alpha_{k+1}^T & q\dfrac{h}{2}\end{bmatrix}\right|$$

Where
$$\alpha_{k+1}=\begin{bmatrix} m_{1,k+1} \\ \vdots \\ m_{k,k+1}\end{bmatrix}$$

By Cholesky, M_{k+1} is said to be positive definite if there exist a lower triangular matrix $\left(L_{i,j}\right)$ such that $M_{k+1}=LL^T$

$$\begin{bmatrix} M_k & \alpha_{k+1} \\ \alpha_{k+1}^T & q\dfrac{h}{2}\end{bmatrix}=\begin{bmatrix} G_k & 0 \\ l_{k+1,1}^T & l_{k+1,k+1}\end{bmatrix}\begin{bmatrix} G_k^T & l_{k+1,1} \\ 0 & l_{k+1,k+1}\end{bmatrix}$$

Where G_k is lower triangular matrix with positive diagonal entries such that $G_kG_k^T=M_k$

$$\begin{bmatrix} M_k & \alpha_{k+1} \\ \alpha_{k+1}^T & q\dfrac{h}{2}\end{bmatrix}=\begin{bmatrix} G_kG_k^T & G_kl_{k+1,1} \\ l_{k+1,1}^TG_k^T & l_{k+1}^Tl_{k+1}+l_{k+1,k+1}^2\end{bmatrix}$$

Thus we have,

$$G_kl_{k+1,1}=\alpha_{k+1}$$

And $l_{k+1,1}^Tl_{k+1,1}+l_{k+1,k+1}^2=q\dfrac{h}{2}$

Hence, since the diagonal entries of G_k are greater than zero, it is nonsingular, so the linear system of equation has a unique solution given by $l_{k+1,1} = G_k^{-1}\alpha_{k+1}$ and a positive value for $l_{k+1,k+1}$ can be obtained

Provided $q\frac{h}{2} - l_{k+1}{}^T l_{k+1} > 0$.

Hence,

$$0 < \det(A_\rho) = \det(\begin{bmatrix} M_k & \alpha_{k+1} \\ \alpha_{k+1}{}^T & q\frac{h}{2} \end{bmatrix}) = \det(M_k)(q\frac{h}{2} - \alpha_{k+1}{}^T (M_k)^{-1}\alpha_{k+1})$$

$$= \det(M_k)[q\frac{h}{2} - (G_k l_{k+1,1})^T (G_k G_k^T)^{-1} G_k l_{k+1,1}]$$

$$= \det(M_k)(q\frac{h}{2} - l_{k+1,1} l_{k+1,1}^T).$$

Since $\det(M_k) > 0,$ it follows that $q\frac{h}{2} - l_{k+1,1}^T l_{k+1,1} > 0$.

Hence $l_{k+1,k+1} = \sqrt{q\frac{h}{2} - l_{k+1,1} l_{k+1,1}^T}$.

We solve the unconstrained minimization equation (2.19) by conjugate gradient algorithm in the inner loop and enforce the feasibility condition in the outer loop as stated in the following Algorithm.

We solve the unconstrained minimization equation (2.19) by conjugate gradient algorithm in the inner loop and enforce the feasibility condition in the outer loop as stated in the following Algorithm.

2.2.1 NUMERICAL ALGORITHM FOR SOLVING QUADRATIC PROGRAMMING PROBLEM

(1) $Choose\ Z_{0,0} \epsilon \underline{R}^{(N+M)S+M}, C > 0, \mu > 0, \lambda > 0, d > 0. Set j = 0$
(2) $Set\ i = 0\ and p_0 = -g_0 = -\nabla L_p(Z_{0,0})$
(3) $Compute\ \alpha_i = \frac{g_i^T g_i}{p_i^T A p_i}$
(4) $Set\ Z_{j,i+1} = Z_{j,i} + \alpha_i p_i$
(5) $Compute\ \nabla L_p(Z_{j,i+1})$
(6) $If\ \nabla L_p(z_{j,i+1}) = 0\ and J Z_{j,i+1} = K, Stop else goto (7)$
(7) $If\ \nabla L_p(z_{j,i+1}) \neq 0, set g_{i+1} = \nabla L_p(Z_{j,i+1})$
$p_{i+1} = -g_{i+1} + \gamma_i p_i$

$$\gamma_i = \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i}$$

(8) $Set \quad i = i+1 \, and \;\, go \; step \; (3)$

(9) $Else, if \quad JZ_{j,i+1} \neq K \quad or \quad JZ_{j,i+1} - K = 0, \; then$

$Set \quad \mu_{k+1} = d\mu_k$

$\lambda_{j+1} = \lambda_j + \mu_j (JZ_j - K)$

(10) $Set \quad j = j+1 \, and \;\, go \; to \; step \; (2)$

What necessitated this algorithm was the problem of ill-condition associated with the control operator A_ρ generated from the non-linear optimization problem via exterior penalty method.

3. Numerical Examples and Presentation of Results

Example 1. Consider the optimal control problem,

$$\min \quad I(x,u) = \int_0^1 (x^2(t) + u^2(t))dt \tag{3.1}$$

Subject to

$$\dot{x}(t) = 5x(t) + x(t-0.3) + 3.021u(t), \quad x(t) = 1 \;\; t \in [-0.3, 0] \tag{3.2}$$

We now present the result of the investigations based on the operator (A_ρ). The results presented here shows the accuracy and the efficiency of the Discretized Continuous Algorithm via Quadratic Programming using augmented lagrangian function to optimal control problem as compared to Discretised Continuous Algorithm via Quadratic Programming using exterior penalty function. Taken $\mu = 1000, h = 0.01$ for both scheme.

Table 1. **Comparison of two methods for Problem 1**

Iterations	Constraints Satisfaction		Objective Value	
	Olotu et al. (2011)	**New Scheme**	**Adekunle (2011)**	**New Scheme**
1	0.1410	0.6995E-1	0.4886	0.7773
2	0.2574E-1	0.1268E-1	0.8141	0.8568
3	0.2811E-2	0.1383E-2	0.8793	0.8726
4	0.2837E-3	0.1395E-3	0.8864	0.8744
5	0.2840E-4	0.1397E-4	0.8872	0.8745
6	0.2840E-5	0.1397E-5	0.8872	0.8746

The objective value using exterior penalty method is 0.8872 while objective value using augmented lagrangian is 0.8746.The effect of the delay term is seen in the objective value using the two schemes as shown in the Table 1 above compared to when without delay term.

Example 2. Consider the optimal control problem,

$$\min \quad I(x,u) = \int_0^1 (x^2(t) + u^2(t))dt \tag{3.3}$$

Subject to

$$\dot{x}(t) = 5x(t) + x(t-0.3) + 3u(t), \quad x(t) = 1 \;\; t \in [-0.3, 0] \tag{3.4}$$

We now present the result of the investigations based on the operator (A_ρ). The results presented here shows the accuracy and the efficiency of the Discretized Continuous Algorithm via Quadratic Programming using augmented lagrangian function to optimal control problem as compared to Discretised Continuous Algorithm via Quadratic Programming using exterior penalty function. Taken $\mu = 1000, h = 0.01$ for both scheme.

Table 2. **Comparison of results using existing scheme and the developed scheme**

Iterations	**Constraints Satisfaction**		**Objective Value**	
	Adekunle (2011)	**New Scheme**	**Adekunle (2011)**	**New Scheme**
1	0.1989	0.9826E-1	0.8247	1.4163
2	0.3768E-1	0.1822E-1	1.4989	1.5773
3	0.4139E-2	0.1993E-2	1.6393	1.6100
4	0.4180E-3	0.2012E-3	1.6548	1.6136
5	0.4184E-4	0.2014E-4	1.6564	1.6140
6	0.4185E-5	0.2014E-5	1.6566	1.6140

The objective value using exterior penalty method is 1.6566 while objective value using augmented lagrangian is 1.6140.The effect of the delay term is seen in the objective value using the two schemes as shown in the Table 2 above compared to when there is no delay term.

4. Conclusions and Recommendation

We have shown that discrete delay optimal control problem can be solved via Conjugate Gradient Method using exterior penalty method and augmented Lagrangian method to construct the control operator A_ρ. However, it is observed that the new scheme gives a better result in terms of accuracy. Hence, it is a better scheme.

Based on the efficiency and robustness of this scheme, we therefore recommend it for delay optimal control problems constrained with partial differential equation. It can also be extended to generalized optimal control problems governed with delay differential equation.

Competing Interests

Authors have declared that no competing interests exist.

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