Hindawi
Abstract and Applied Analysis
Volume 2023, Article ID 5572893, 10 pages
https://doi.org/10.1155/2023/5572893

Research Article

Generalized Enriched Nonexpansive Mappings and Their Fixed Point Theorems

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Received 5 April 2023; Revised 3 June 2023; Accepted 15 November 2023; Published 2 December 2023

Academic Editor: Paul Eloe

This paper introduces a novel category of nonlinear mappings and provides several theorems on their existence and convergence in Banach spaces, subject to various assumptions. Moreover, we obtain convergence theorems concerning iterates of α-Krasnosel'skiĭ mapping associated with the newly defined class of mappings. Further, we present that α-Krasnosel'skiĭ mapping associated with b-enriched quasinonexpansive mapping is asymptotically regular. Furthermore, some new convergence theorems concerning b-enriched quasinonexpansive mappings have been proved.

1. Introduction

Consider a Banach space $(\mathscr{X}, \|.\|)$ and a nonempty subset $\mathscr{B} \subseteq \mathscr{X}$. A mapping $\Psi : \mathscr{B} \to \mathscr{B}$ is nonexpansive if $\|\Psi(\vartheta) - \Psi(\varrho)\| \le \|\vartheta - \varrho\|$ for all $\vartheta, \varrho \in \mathscr{B}$. A point $z \in \mathscr{B}$ is a fixed point of Ψ if $\Psi(z) = z$. While nonexpansive mappings may not have fixed points in a general Banach space. Browder [1], Göhde [2], and Kirk [3] independently proved fixed point theorems for nonexpansive mappings with certain geometric properties, such as uniform convexity or normal structure. Since then, numerous authors have obtained various extensions and generalizations of nonexpansive mappings and their results. Some of these notable extensions and generalizations are summarized in Pant et al.'s [4] study.

In 2008, Suzuki [5] introduced a novel category of nonexpansive mappings called mappings that fulfill condition (C) and derived significant fixed point results for this category. García-Falset et al. [6] further generalized condition (C) into the class of mappings satisfying condition (E).

Definition 1 [6]. Let $\mathscr{B}$ be a nonempty subset of a Banach space $\mathscr{X}$. A mapping $\Psi : \mathscr{B} \to \mathscr{B}$ is said to satisfy condition (E_μ) on $\mathscr{B}$ if there exists $\mu \ge 1$ such that for all $\vartheta, \varrho \in \mathscr{B}$:

$$\|\vartheta - \Psi(\varrho)\| \le \mu\|\vartheta - \Psi(\vartheta)\| + \|\vartheta - \varrho\|. \tag{1}$$

We say that Ψ satisfies condition (E) on $\mathscr{B}$ whenever Ψ satisfies (E_μ) for some $\mu \ge 1$.

This class of mappings properly contains many important classes of generalized nonexpansive mappings, see Pant et al's [7] study.

A novel category of nonlinear mappings was introduced by Berinde [8] in a recent publication.

Definition 2 [8]. Let $(\mathscr{X}, \|.\|)$ be a Banach space. A mapping $\Psi : \mathscr{X} \to \mathscr{X}$ is said to be b-enriched nonexpansive mapping if $\exists\, b \in [0, \infty)$ such that:

$$\|b(\zeta - \varrho) + \Psi(\vartheta) - \Psi(\varrho)\| \le (b+1)\|\vartheta - \varrho\| \;\forall\, \vartheta, \varrho \in \mathscr{X}. \tag{2}$$

In the recent years, a number of papers have appeared in the literature dealing with the fixed point theorems for enriched nonexpansive type mappings [9–11]. It was proven that any enriched contraction mapping defined on a Banach space has a unique fixed point, which can be approximated by means of the Krasnoselskij iterative scheme. In Berinde's

[12] study (also [13]), it was demonstrated that every nonexpansive mapping Ψ is a zero-enriched mapping. Building in this work, Shukla and Pant [14] recently extended the category of enriched nonexpansive mappings in the vein of Suzuki and introduced the subsequent class of mappings.

Definition 3. Let $(\mathscr{X}, \|.\|)$ be a Banach space and $\mathscr{B}$ a nonempty subset of $\mathscr{X}$. A mapping $\Psi : \mathscr{B} \to \mathscr{B}$ is said to be Suzuki-enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that for all $\vartheta, \varrho \in \mathscr{B}$:

$$\begin{aligned} &\frac{1}{2(b+1)}\|\vartheta - \Psi(\vartheta)\| \le \|\vartheta - \varrho\| \text{ implies} \\ &\|b(\zeta - \varrho) + \Psi(\vartheta) - \Psi(\varrho)\| \le (b+1)\|\vartheta - \varrho\|. \end{aligned} \tag{3}$$

It can be seen that every Suzuki-nonexpansive mapping Ψ is a Suzuki-enriched nonexpansive mapping with $b = 0$. Motivated by García-Falset et al. [6], we generalize Suzuki-enriched nonexpansive mappings and consider a new class of mappings known as (E)-enriched nonexpansive mappings. In fact, we introduce a class of mapping, which contains both the Suzuki-enriched nonexpansive mappings and the class of mappings satisfying condition (E). Indeed the class of b-enriched nonexpansive mappings and that of mappings satisfying condition (E) are independent in nature. A couple of examples below illustrate these facts.

Example 1 [4]. Consider the mapping $\Psi : \mathscr{B} \to \mathscr{B}$ defined by $\Psi(\vartheta) = \frac{1}{\vartheta}$, where $\mathscr{B} = [\frac{1}{2}, 2] \subset \mathbb{R}$. The set of fixed point of Ψ is $\{1\}$, and we can say that $F(\Psi) = \{1\}$. Moreover, Ψ is a nonexpansive mapping that satisfies the $\frac{3}{2}$-enriched condition. However, at $\vartheta = 1$ and $\varrho = \frac{1}{2}$, Ψ fails to satisfy condition (E).

Example 2 [5]. Let $\mathscr{B} = [0, 3] \subset \mathbb{R}$ with the usual norm. Define $\Psi : \mathscr{B} \to \mathscr{B}$ by the following equation:

$$\Psi(\vartheta) = \begin{cases} 0, & \text{if } \zeta \ne 3, \\ 1, & \text{if } \zeta = 3. \end{cases} \tag{4}$$

Then, Ψ satisfies condition (E). However at $\vartheta = 2.5$ and $\varrho = 3$, Ψ is not b-enriched nonexpansive mapping for any $b \in [0, \infty)$.

This paper is organized as follows: Section 2 deals with some preliminary results which are utilized throughout this paper. In Section 3, we coined a new class of mappings, namely, (E)-enriched nonexpansive mapping. We show that (E)-enriched nonexpansive mapping properly contains some nonlinear mappings and present an illustrative example. Section 4 is devoted to some existence and convergence theorems concerning (E)-enriched nonexpansive mapping. In Section 5, we present some new developments of enriched quasinonexpansive mapping. Particularly, an (E)-enriched nonexpansive mapping with fixed point is an enriched quasinonexpansive mapping. We discuss convergence of the iterates of α-Krasnosel'skiĭ mapping associated with enriched quasinonexpansive mapping.

2. Preliminaries

We denote $F(\Psi)$ the set of all fixed points of mapping Ψ, i.e., $F(\Psi) = \{z \in \mathscr{B} : \Psi(z) = z\}$. A Banach space $\mathscr{X}$ is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, $\exists\, \delta > 0$ such that $\|\frac{\zeta+\varrho}{2}\| \le 1 - \delta$ for all $\zeta, \varrho \in \mathscr{X}$ with $\|\zeta\| = \|\varrho\| = 1$ and $\|\zeta - \varrho\| > \varepsilon$. A Banach space $\mathscr{X}$ is strictly convex if:

$$\left\|\frac{\zeta+\varrho}{2}\right\| < 1, \tag{5}$$

whenever $\zeta, \varrho \in \mathscr{X}$ with $\|\zeta\| = \|\varrho\| = 1$, $\zeta \ne \varrho$ [15].

Theorem 1 [15]. *Let $\mathscr{X}$ be a uniformly convex Banach space. Then for any $d > 0$, $\varepsilon > 0$, and $\zeta, \varrho \in \mathscr{X}$ with $\|\zeta\| \le d$, $\|\varrho\| \le d$, $\|\zeta - \varrho\| \ge \varepsilon$, there exists a $\delta > 0$ such that:*

$$\left\|\frac{1}{2}(\zeta + \varrho)\right\| \le \left[1 - \delta\left(\frac{\varepsilon}{d}\right)\right] d. \tag{6}$$

Definition 4 [16]. A Banach space $\mathscr{X}$ satisfies Opial property if for every weakly convergent sequence $\{\zeta_n\}$ with weak limit $\zeta \in \mathscr{X}$, it holds:

$$\liminf_{n\to\infty} \|\zeta_n - \zeta\| < \liminf_{n\to\infty} \|\zeta_n - \varrho\|, \tag{7}$$

for all $\varrho \in \mathscr{X}$ with $\zeta \ne \varrho$.

Definition 5 [17]. Suppose $\mathscr{B}$ is a nonempty subset of a Banach space $\mathscr{X}$. Let ζ be an element in $\mathscr{X}$ such that there exists a point ϱ in $\mathscr{B}$ satisfying the following condition: for any $z \in \mathscr{B}$, $\|\varrho - \zeta\| \le \|z - \zeta\|$. In this case, we refer to ϱ as a metric projection of ζ onto $\mathscr{B}$ and denote it by $P_{\mathscr{B}}(\zeta)$. If $P_{\mathscr{B}}(\zeta)$ exists and is uniquely determined for all $x \in \mathscr{X}$, then we call the mapping $P_{\mathscr{B}} : \mathscr{X} \to \mathscr{B}$, the metric projection onto $\mathscr{B}$.

Definition 6 [17]. A mapping $\Psi : \mathscr{B} \to \mathscr{B}$ is said to be quasinonexpansive if:

$$\|\Psi(\zeta) - z\| \le \|\zeta - z\|, \tag{8}$$

for all $\zeta \in \mathscr{B}$ and $z \in F(\Psi)$.

The fact that a nonexpansive mapping with a fixed point is quasinonexpansive is widely recognized. However, it should be noted that the converse may not hold.

Proposition 1 [6]. *Let $\mathscr{B}$ be a nonempty subset of a Banach space $\mathscr{X}$. If $\Psi : \mathscr{B} \to \mathscr{B}$ is a mapping satisfying condition (E) with $F(\Psi) \ne \emptyset$ then Ψ is quasinonexpansive.*

Definition 7 [18]. Let $\mathscr{B}$ be a nonempty convex subset of a Banach space $\mathscr{X}$ and $\Psi : \mathscr{B} \to \mathscr{B}$ be a mapping. A mapping $\Psi_\alpha : \mathscr{B} \to \mathscr{B}$ is said to be an α-Krasnosel'skiĭ mapping associated with Ψ if there exists $\alpha \in (0,1)$ such that:

$$\Psi_\alpha(\zeta) = (1-\alpha)\zeta + \alpha\Psi(\zeta), \tag{9}$$

for all $\zeta \in \mathscr{B}$.

Definition 8 [19]. Let $\mathscr{B}$ be a nonempty subset of a Banach space $\mathscr{X}$. A mapping $\Psi : \mathscr{B} \to \mathscr{B}$ is called asymptotically regular if:

$$\lim_{n\to\infty} \|\Psi^n(\zeta) - \Psi^{n+1}(\zeta)\| = 0. \tag{10}$$

Lemma 1 (Browder [20]; demiclosedness principle). *Suppose $\mathscr{B}$ is a nonempty subset of a Banach space $\mathscr{X}$ that satisfies the Opial property. Let $\Psi : \mathscr{B} \to \mathscr{B}$ be a mapping that satisfies condition (E). Consider a sequence ζ_n in $\mathscr{B}$ such that ζ_n weakly converges to ζ, and $\lim_{n\to\infty} \|\zeta_n - \Psi(\zeta_n)\| = 0$. Then, we have $\Psi(\zeta) = \zeta$, which implies that $I - \Psi$ is demiclosed at zero.*

Proof. The proof directly follows from [6, Theorem 1]. □

Lemma 2 (Berinde [8]). *Let $\mathscr{B}$ be a nonempty convex subset of a Banach space $\mathscr{X}$, and $\Psi : \mathscr{B} \to \mathscr{B}$ be a mapping. Define $S : \mathscr{B} \to \mathscr{B}$ as follows:*

$$S(\zeta) = (1-\lambda)\zeta + \lambda\Psi(\zeta), \tag{11}$$

for all $\zeta \in \mathscr{B}$ and $\lambda \in (0,1)$. Then, $F(S) = F(\Psi)$.

3. (E)-Enriched Nonexpansive Mapping

This section presents a novel category of mappings, which we describe as follows.

Definition 9. Let $(\mathscr{X}, \|.\|)$ be a Banach space and $\mathscr{B}$ be a nonempty subset of $\mathscr{X}$. We define a mapping $\Psi : \mathscr{B} \to \mathscr{X}$ an (E)-enriched nonexpansive mapping if $\exists\ b \in [0,\infty)$ and $M \in [1,\infty)$ such that:

$$\begin{aligned}\|b(\zeta-\varrho) + \zeta - \Psi(\varrho)\| &\le M\|\zeta - \Psi(\zeta)\| \\ &\quad + (b+1)\|\zeta - \varrho\|\ \forall\, \zeta, \varrho \in \mathscr{B}.\end{aligned} \tag{12}$$

It can be seen that every mapping Ψ satisfying condition (E) is an (E)-enriched nonexpansive mapping with $b = 0$.

Remark 1. If $\mathscr{B}$ is a nonempty subset of $\mathscr{X}$ and $\Psi : \mathscr{B} \to \mathscr{X}$ is (E)-enriched nonexpansive, and there exists a sequence $\{\zeta_n\}$ in $\mathscr{B}$ such that $\|\zeta_n - \Psi(\zeta_n)\| \to 0$. Such a sequence is called almost fixed point sequence (a.f.p.s.) for Ψ.

Proposition 2. *Let $\Psi : \mathscr{B} \to \mathscr{X}$ be a Suzuki-enriched nonexpansive mapping with any $b \in [0,\infty)$. Then, Ψ is an (E)-enriched nonexpansive mapping for any $b \in [0,\infty)$ and $M = 2b + 3$.*

Proof. We assume that Ψ is a Suzuki-enriched nonexpansive mapping. Then, $\frac{1}{2(b+1)}\|\zeta - \Psi(\zeta)\| \le \|\zeta - \Psi(\zeta)\|$ implies the following equation:

$$\|b(\zeta - \Psi(\zeta)) + \Psi(\zeta) - \Psi^2(\zeta)\| \le (b+1)\|\zeta - \Psi(\zeta)\|. \tag{13}$$

Now, we show that either:

$$\frac{1}{2(b+1)}\|\zeta - \Psi(\zeta)\| \le \|\zeta - \varrho\| \text{ or } \frac{1}{2(b+1)}\|\Psi(\zeta) - \Psi^2(\zeta)\| \le \|\Psi(\zeta) - \varrho\|. \tag{14}$$

Arguing by contradiction, we suppose that:

$$\frac{1}{2(b+1)}\|\zeta - \Psi(\zeta)\| > \|\zeta - \varrho\| \text{ and } \frac{1}{2(b+1)}\|\Psi(\zeta) - \Psi^2(\zeta)\| > \|\Psi(\zeta) - \varrho\|. \tag{15}$$

By the triangle inequality, we get the following equation:

$$\begin{aligned}\|\zeta - \Psi(\zeta)\| &\le \|\zeta - \varrho\| + \|\Psi(\zeta) - \varrho\| \\ &< \frac{1}{2(b+1)}\|\zeta - \Psi(\zeta)\| + \frac{1}{2(b+1)}\|\Psi(\zeta) - \Psi^2(\zeta)\| \\ &= \frac{1}{2(b+1)}\|\zeta - \Psi(\zeta)\| + \frac{1}{2(b+1)}[\|b\zeta - b\zeta + b\Psi(\zeta) \\ &\quad - b\Psi(\zeta) + \Psi(\zeta) - \Psi^2(\zeta)\|] \\ &\le \frac{1}{2(b+1)}\|\zeta - \Psi(\zeta)\| + \frac{1}{2(b+1)}\|b\zeta - b\Psi(\zeta) + \Psi(\zeta) - \Psi^2(\zeta)\| \\ &\quad + \frac{1}{2(b+1)}\|b\zeta - b\Psi(\zeta)\|.\end{aligned} \tag{16}$$

By Equation (13), we get the following equation:

$$\begin{aligned}\|\zeta - \Psi(\zeta)\| &< \frac{1}{2(b+1)}\|\zeta - \Psi(\zeta)\| + \frac{1}{2(b+1)}(b+1)\|\zeta - \Psi(\zeta)\| \\ &+ \frac{1}{2(b+1)} b\|\zeta - \Psi(\zeta)\| \\ &= \|\zeta - \Psi(\zeta)\|,\end{aligned} \tag{17}$$

which is a contradiction. Consequently, Equation (14) holds. Therefore, from Equation (14), we have either:

$$\begin{aligned}&\|b(\zeta - \varrho) + \Psi(\zeta) - \Psi(\varrho)\| \le (b+1)\|\zeta - \varrho\|, \\ &\text{or} \\ &\|b(\Psi(\zeta) - \varrho) + \Psi^2(\zeta) - \Psi(\varrho)\| \le (b+1)\|\Psi(\zeta) - \varrho\|.\end{aligned} \tag{18}$$

In the first case, we have the following equation:

$$\begin{aligned}\|b(\zeta - \varrho) + \zeta - \Psi(\varrho)\| &\le \|b(\zeta - \varrho) + \Psi(\zeta) - \Psi(\varrho)\| \\ &+ \|\zeta - \Psi(\zeta)\| \\ &\le \|\zeta - \Psi(\zeta)\| + (b+1)\|\zeta - \varrho\|.\end{aligned} \tag{19}$$

In the other case, we have the following equation:

$$\begin{aligned}\|b(\zeta - \varrho) + \zeta - \Psi(\varrho)\| &\le \|b(\Psi(\zeta) - \varrho) + \Psi^2(\zeta) - \Psi(\varrho) + b\zeta - b\Psi(\zeta) + \zeta - \Psi^2(\zeta)\| \\ &\le \|b(\Psi(\zeta) - \varrho) + \Psi^2(\zeta) - \Psi(\varrho)\| + \|b\zeta - b\Psi(\zeta) + \zeta - \Psi^2(\zeta)\| \\ &\le \|b(\Psi(\zeta) - \varrho) + \Psi^2(\zeta) - \Psi(\varrho)\| + \|b(\zeta - \Psi(\zeta)) + \Psi(\zeta) - \Psi^2(\zeta)\| \\ &+ \|\zeta - \Psi(\zeta)\| \\ &\le (b+1)\|\Psi(\zeta) - \varrho\| + (b+1)\|\zeta - \Psi(\zeta)\| + \|\zeta - \Psi(\zeta)\| \\ &\le (b+1)\|\zeta - \Psi(\zeta)\| + (b+1)\|\zeta - \varrho\| + (b+1)\|\zeta - \Psi(\zeta)\| \\ &+ \|\zeta - \Psi(\zeta)\| \\ &= (2b+3)\|\zeta - \Psi(\zeta)\| + (b+1)\|\zeta - \varrho\|.\end{aligned} \tag{20}$$

Therefore in both the cases, we get the following equation:

$$\|b(\zeta - \varrho) + \zeta - \Psi(\varrho)\| \le (2b+3)\|\zeta - \Psi(\zeta)\| + (b+1)\|\zeta - \varrho\|. \tag{21}$$

□

The following example shows that (E)-enriched nonexpansive mapping properly contains the class of Suzuki-enriched nonexpansive mappings.

Example 3. Let $\mathbb{R}$ be the set of real numbers equipped with the standard norm and $\mathscr{B} = [0, 4]$ a subset of $\mathbb{R}$. Let $\Psi : \mathscr{B} \to \mathscr{B}$ be a mapping defined by the following equation:

$$\Psi(\zeta) = \begin{cases} 0, & \text{if } \zeta \neq 4, \\ 3, & \text{if } \zeta = 4. \end{cases} \tag{22}$$

First, we show that Ψ is (E)-enriched nonexpansive mapping. For this, we consider the following nontrivial cases:

Case (1). If $\zeta \le 3$ and $\varrho = 4$, then the following equation is obtained:

$$\begin{aligned}\|b(\zeta - \varrho) + \zeta - \Psi(\varrho)\| = \|b(\zeta - 4) + (\zeta - 3)\| &\le \|b(\zeta - 4)\| + \|\zeta - 3\| \\ &\le \|\zeta\| + (b+1)\|4 - \zeta\| \\ &= \|\zeta - \Psi(\zeta)\| + (b+1)\|\varrho - \zeta\|.\end{aligned} \tag{23}$$

Case (2). If $\zeta > 3$ and $\varrho = 4$, then the following equation is obtained:

$$\begin{aligned}\|b(\zeta - \varrho) + \zeta - \Psi(\varrho)\| = \|b(\zeta - 4) + (\zeta - 3)\| &\le b\|\zeta - 4\| + \|\zeta - 3\| \\ &\le \|\zeta\| + (b+1)\|4 - \zeta\| \\ &\le \|\zeta - \Psi(\zeta)\| + (b+1)\|\varrho - \zeta\|.\end{aligned} \tag{24}$$

Case (3). If $\zeta = 4$ and $\varrho \neq 4$, then the following equation is obtained:

$$\|b(\zeta - \varrho) + \zeta - \Psi(\varrho)\| = \|b(4 - \varrho) + 4\| \leq 4 + b\|4 - \varrho\| \leq 4\|\zeta - \Psi(\zeta)\| + (b+1)\|\varrho - \zeta\|. \tag{25}$$

Moreover, for $\zeta = 3$ and $\varrho = 4$, we get the following equations:

$$\frac{1}{2(b+1)}\|4 - \Psi(4)\| = \frac{1}{2(b+1)} \leq 1 = \|\zeta - \varrho\|, \tag{26}$$

and

$$\|b(\zeta - \varrho) + \Psi(\zeta) - \Psi(\varrho)\| = |b(3-4) + (0-3) = |-b - 3| = b + 3 > (b+1) = (b+1)\|\zeta - \varrho\|. \tag{27}$$

Thus, Ψ is not a Suzuki-enriched nonexpansive mapping with any $b \in [0, \infty)$.

4. Some Fixed Point Theorems

In this section, we present some new fixed point theorems for (E)-enriched nonexpansive mappings.

Theorem 2. *Let $(\mathcal{X}, \|.\|)$ be a Banach space and $\mathcal{B}$ a nonempty subset of $\mathcal{X}$. Let $\Psi : \mathcal{B} \to \mathcal{X}$ be a mapping. If:*

(a) *Ψ is an (E)-enriched nonexpansive mapping on $\mathcal{B}$,*
(b) *there exists an a.f.p.s. $\{\zeta_n\}$ for Ψ in $\mathcal{B}$ such that $\{\zeta_n\}$ converges weakly to a point z in $\mathcal{B}$, and*
(c) *$(\mathcal{X}, \|.\|)$ satisfies the Opial property.*

Then, $\Psi(z) = z$.

Proof. By the definition of mapping Ψ, we have the following equation:

$$\|b(\zeta - \varrho) + \zeta - \Psi(\varrho)\| \leq M\|\zeta - \Psi(\zeta)\| + (b+1)\|\zeta - \varrho\|, \tag{28}$$

for all $\zeta, \varrho \in \mathcal{B}$. Take, $\mu = \frac{1}{b+1} \in (0,1)$ and put $b = \frac{1-\mu}{\mu}$ in Equation (28), then the above inequality is equivalent to the following equations:

$$\|(1-\mu)(\zeta - \varrho) + \mu(\zeta - \Psi(\varrho))\| \leq M\mu\|\zeta - \Psi(\zeta)\| + \|\zeta - \varrho\|, \tag{29}$$

and

$$\|\zeta - ((1-\mu)\varrho + \mu\Psi(\varrho))\| \leq M\mu\|\zeta - \Psi(\zeta)\| + \|\zeta - \varrho\|. \tag{30}$$

Define the mapping S as follows:

$$S(\zeta) = (1-\mu)\zeta + \mu\Psi(\zeta) \text{ for all } \zeta \in \mathcal{B}. \tag{31}$$

Thus, the following equation is obtained:

$$\|S(\zeta) - \zeta\| = \mu\|\Psi(\zeta) - \zeta\|. \tag{32}$$

Then, from Equation (30), we get the following equation:

$$\|\zeta - S(\varrho)\| \leq M\|\zeta - S(\zeta)\| + \|\zeta - \varrho\|, \tag{33}$$

for all $\zeta, \varrho \in \mathcal{B}$. Thus, S is a mapping satisfying condition (E). From Equation (32), it follows that if $\{\zeta_n\}$ is an a.f.p.s. for Ψ then $\{\zeta_n\}$ is an a.f.p.s. for S. Thus, all the assumptions of [6, Theorem 1] are satisfied and S has a fixed point in $\mathcal{B}$, that is, $z \in F(S)$. From Lemma 2, $F(S) = F(\Psi) \neq \emptyset$. This completes the proof. □

Corollary 1. *Let $(\mathcal{X}, \|.\|)$ be a Banach space and $\mathcal{B}$ a nonempty weakly compact subset of a Banach space $\mathcal{X}$. Suppose that $(\mathcal{X}, \|.\|)$ satisfies the Opial property. Let $\Psi : \mathcal{B} \to \mathcal{X}$ be an (E)-enriched nonexpansive mapping on $\mathcal{B}$. Then, Ψ has a fixed point in $\mathcal{B}$ if and only if Ψ admits an a.f.p.s.*

Theorem 3. *Let $(\mathcal{X}, \|.\|)$ be a Banach space and $\mathcal{B}$ a nonempty compact subset of a Banach space $\mathcal{X}$. Let $\Psi : \mathcal{B} \to \mathcal{X}$ be an (E)-enriched nonexpansive mapping on $\mathcal{B}$. Then, Ψ has a fixed point in $\mathcal{B}$ if and only if Ψ admits an a.f.p.s.*

Proof. By following the proof of Theorem 1, we can construct a mapping S that satisfies condition (E). Therefore, all the conditions of [6, Theorem 2] are satisfied, and we can guarantee that S has at least one fixed point. Using Lemma 2, we can conclude that $F(S) = F(\Psi) \neq \emptyset$. This completes the proof. □

Theorem 4. *Let $(\mathcal{X}, \|.\|)$ be a uniformly convex in every direction (or UCED) Banach space and $\mathcal{B}$ a nonempty weakly compact convex subset of $\mathcal{X}$.* Let *$\Psi : \mathcal{B} \to \mathcal{X}$ be a mapping. If:*

(a) Ψ *is an (E)-enriched nonexpansive mapping on* $\mathscr{B}$, *and*

(b) $\inf\{||\zeta - \Psi(\zeta)|| : \zeta \in \mathscr{B}\} = 0$,

then Ψ admits a fixed point.

Proof. In view of Theorem 1 and [6, Theorem 3], one can complete the proof. □

In the next theorem, we present the convergence of iterates of α-Krasnosel'skiĭ mapping associated with (E)-enriched nonexpansive mapping.

Theorem 5. *Let* $\mathscr{X}$ *be a Banach space and* $\Psi : \mathscr{X} \to \mathscr{X}$ *be an (E)-enriched nonexpansive mapping. For a given* $\zeta_0 \in \mathscr{X}$ *and* $\alpha \in (0, \frac{1}{b+1})$, *if the sequence of iterates* $\{\Psi_\alpha^n(\zeta_0)\}$ *converges strongly to* $\zeta^\dagger$, *where* Ψ_α *is the α-Krasnosel'skiĭ mapping associated with* Ψ, *then* $\zeta^\dagger \in F(\Psi)$.

Proof. Using the same technique as in Theorem 1, one can define a mapping $S : \mathscr{X} \to \mathscr{X}$ as follows:

$$S(\zeta) = \left(1 - \frac{1}{b+1}\right)\zeta + \frac{1}{b+1}\Psi(\zeta) \text{ for all } \zeta \in \mathscr{X}, \quad (34)$$

and S is a mapping satisfying condition (E). For a given $\zeta_0 \in \mathscr{X}$ and $\gamma \in (0, 1)$, one can define a sequence $\{\zeta_n\}$ as follows:

$$\zeta_n = S_\gamma^n(\zeta_0) = (1-\gamma)\zeta_{n-1} + \gamma S(\zeta_{n-1}) \text{ for all } n \in \mathbb{N}. \quad (35)$$

Using the definition of S, we have the following equation

$$\zeta_n = (1-\gamma)\zeta_{n-1} + \gamma S(\zeta_{n-1}) = \left(1 - \gamma\frac{1}{b+1}\right)\zeta_{n-1} + \gamma\frac{1}{b+1}\Psi(\zeta_{n-1}). \quad (36)$$

Take $\alpha = \frac{\gamma}{b+1} \in (0, \frac{1}{b+1})$, then:

$$\zeta_n = (1-\alpha)\zeta_{n-1} + \alpha\Psi(\zeta_{n-1}). \quad (37)$$

Since $\{\zeta_n\} = \{\Psi_\alpha^n(\zeta_0)\}$ strongly converges to a point $\zeta^\dagger$ in $\mathscr{X}$. Thus, all assumptions of [21, Theorem 1] are satisfied, and $\zeta^\dagger \in F(S)$. But $F(S) = F(\Psi)$. This completes the proof. □

Theorem 6. *Let* $\mathscr{B}$ *be a nonempty closed convex subset of a uniformly convex Banach space* $\mathscr{X}$ *and* $\Psi : \mathscr{B} \to \mathscr{B}$ *an (E)-enriched nonexpansive mapping with* $F(\Psi) = \{\zeta^\dagger\}$. *Assume that the mapping* $I - \Psi$ *is demiclosed at zero. Then for each* $\zeta_0 \in \mathscr{B}$, *the sequence of iterates* $\{\Psi_\alpha^n(\zeta_0)\}$ *converges weakly to* $\zeta^\dagger$, *where* Ψ_α *is the α-Krasnosel'skiĭ mapping associated with* Ψ *and* $\alpha \in (0, \frac{1}{b+1})$.

Proof. Using the same technique as in Theorem 1, one can define a mapping $S : \mathscr{B} \to \mathscr{B}$ such that S is a mapping satisfying condition (E). In fact, the demiclosedness of $I - \Psi$ and $I - \Psi_\alpha$ at zeros are equivalent. Further, demiclosedness of $I - \Psi$ and $I - S$ at zeros are equivalent. Keeping [21, Theorem 2.2] in mind, one can complete the proof. □

Theorem 7. *Let* $\mathscr{B}$ *be a nonempty closed convex subset of a uniformly convex Banach space* $\mathscr{X}$ *which has the Opial property. Let* $\Psi : \mathscr{B} \to \mathscr{B}$ *be an (E)-enriched nonexpansive mapping with* $F(\Psi) \neq \emptyset$. *Then for each* $\zeta_0 \in \mathscr{B}$ *and* $\alpha \in (0, \frac{1}{b+1})$, *the sequence of iterates* $\{\Psi_\alpha^n(\zeta_0)\}$ *converges weakly to a fixed point of* Ψ.

Proof. From [21, Theorem 2.2] and Theorem 6, one can complete the proof. □

5. Enriched Quasinonexpansive Mapping

In this section, we present some new convergence results for b-enriched quasinonexpansive mapping. Shukla and Pant [14] introduced the following new class of mappings.

Definition 10. Let $(\mathscr{X}, \|.\|)$ be a Banach space and $\mathscr{B}$ a nonempty subset of $\mathscr{X}$. A mapping $\Psi : \mathscr{B} \to \mathscr{B}$ is said to be b-enriched quasinonexpansive mapping if there exists $b \in [0, \infty)$ such that for all $\zeta \in \mathscr{B}$ and $\varrho \in F(\Psi) \neq \emptyset$:

$$\|b(\zeta - \varrho) + \Psi(\zeta) - \varrho\| \le (b+1)\|\zeta - \varrho\|. \quad (38)$$

It is noted that every quasinonexpansive mapping is a zero-enriched quasinonexpansive mapping and every b-enriched nonexpansive mapping with a fixed point is b-enriched quasinonexpansive mapping.

Proposition 3. *Let* $\Psi : \mathscr{B} \to \mathscr{B}$ *be an (E)-enriched nonexpansive mapping with any* $b \in [0, \infty)$, $M \in [1, \infty)$, *and* $F(\Psi) \neq \emptyset$. *Then,* Ψ *is a b-enriched quasinonexpansive mapping for any* $b \in [0, \infty)$.

Proof. Let $\varrho \in F(\Psi)$ and $\zeta \in \mathscr{B}$, we have the following equation:

$$\begin{aligned} \|b(\zeta - \varrho) + \Psi(\zeta) - \varrho\| &= \|-\{b(\varrho - \zeta) + \varrho - \Psi(\zeta)\}\| \\ &= \|b(\varrho - \zeta) + \varrho - \Psi(\zeta)\| \\ &\le M\|\varrho - \Psi(\varrho)\| + (b+1)\|\varrho - \zeta\| \\ &= (b+1)\|\varrho - \zeta\|. \end{aligned} \quad (39)$$

Thus, Ψ is a b-enriched quasinonexpansive mapping. □

The following example demonstrates that converse of the above proposition does not hold.

Example 4 [6]. Let $\mathscr{B} = [-1, 1]$ be a subset of $\mathbb{R}$. Let $\Psi : \mathscr{B} \to \mathscr{B}$ be a mapping defined by the following equation:

$$\Psi(\zeta)=\begin{cases}\dfrac{\zeta}{1+|\zeta|}\sin\left(\dfrac{1}{\zeta}\right), & \text{if } \zeta\neq 0,\\ 0, & \text{if } \zeta=0.\end{cases} \tag{40}$$

Clearly, $F(\Psi)=\{0\}\neq\emptyset$. It can be seen that:

$$|\Psi(\zeta)|\leq|\zeta|, \tag{41}$$

for all $\zeta\in[-1,1]$ and Ψ is a b-enriched quasinonexpansive mapping with $b=0$.

On the other hand, let $\zeta_n=\frac{1}{2n\pi+\frac{\pi}{2}}$ and $\varrho_n=\frac{1}{2n\pi}$ for all $n\in\mathbb{N}$, we get the following equation:

$$\begin{aligned}&\frac{|b(\zeta_n-\varrho_n)+\zeta_n-\Psi(\varrho_n)|-(b+1)|\zeta_n-\varrho_n|}{|\zeta_n-\Psi(\zeta_n)|}\\ &=\frac{|(b+1)\zeta_n-b\varrho_n|-(b+1)(\varrho_n-\zeta_n)}{|\zeta_n-\Psi(\zeta_n)|}\\ &=\frac{|(b+1)\zeta_n-b\varrho_n|-(b+1)(\varrho_n-\zeta_n)}{\frac{\zeta_n^2}{1+\zeta_n}}\\ &=\frac{(1+\zeta_n)}{\zeta_n}\left[\left|(b+1)-b\frac{\varrho_n}{\zeta_n}\right|-(b+1)\left(\frac{\varrho_n}{\zeta_n}-1\right)\right]\rightarrow\infty,\end{aligned} \tag{42}$$

as $n\rightarrow\infty$. Hence, Ψ is not an (E)-enriched nonexpansive mapping for any $b\in[0,\infty)$.

Theorem 8. *Let $\mathscr{B}$ be a nonempty convex subset of a uniformly convex Banach space $\mathscr{X}$. If $\Psi:\mathscr{B}\rightarrow\mathscr{B}$ is a b-enriched quasinonexpansive mapping with $F(\Psi)\neq\emptyset$, Then, the α-Krasnosel'skiĭ mapping Ψ_α for $\alpha\in(0,\frac{1}{b+1})$ is asymptotically regular.*

Proof. By the definition of b-enriched quasinonexpansive mapping, we have the following equation:

$$\|b(\zeta-\varrho)+\Psi(\zeta)-\varrho\|\leq(b+1)\|\zeta-\varrho\|\ , \tag{43}$$

for all $\zeta\in\mathscr{B}$ and $\varrho\in F(\Psi)$. Take $\mu=\frac{1}{b+1}\in(0,1)$ and put $b=\frac{1-\mu}{\mu}$ in Equation (43), then the above inequality is equivalent to the following equation:

$$\|(1-\mu)(\zeta-\varrho)+\mu(\Psi(\zeta)-\varrho)\|\leq\|\zeta-\varrho\|. \tag{44}$$

Define the mapping S as follows:

$$S(\zeta)=(1-\mu)\zeta+\mu\Psi(\zeta)\ \text{for all}\ \zeta\in\mathscr{B}. \tag{45}$$

From Lemma 2, $F(S)=F(\Psi)$. Then, from Equation (44), we get the following equation:

$$\|S(\zeta)-\varrho\|\leq\|\zeta-\varrho\|, \tag{46}$$

for all $\zeta\in\mathscr{B}$ and $\varrho\in F(S)$. Thus, $S:\mathscr{B}\rightarrow\mathscr{B}$ is a quasinonexpansive mapping. And, we obtain the following equation:

$$\|S(\zeta)-\zeta\|=\mu\|\Psi(\zeta)-\zeta\|. \tag{47}$$

Let $\varrho_0\in\mathscr{B}$. For each $n\in\mathbb{N}\cup\{0\}$, define $\varrho_{n+1}=\Psi_\alpha(\varrho_n)$. Thus, the following equations are obtained:

$$\Psi_\alpha(\varrho_n)=\varrho_{n+1}=(1-\alpha)\varrho_n+\alpha\Psi(\varrho_n), \tag{48}$$

and

$$\Psi_\alpha(\varrho_n)-\varrho_n=\Psi_\alpha(\varrho_n)-\Psi_\alpha(\varrho_{n-1})=\alpha(\Psi(\varrho_n)-\varrho_n). \tag{49}$$

In order to prove that Ψ_α is asymptotically regular, it suffices to prove that $\lim_{n\rightarrow\infty}\|\Psi(\varrho_n)-\varrho_n\|=0$. Since $F(\Psi)\neq\emptyset$, let $\zeta_0\in F(\Psi)$. Since S is a quasinonexpansive mapping and $F(S)=F(\Psi)$:

$$\|\zeta_0-S(\varrho_n)\|\leq\|\zeta_0-\varrho_n\|, \tag{50}$$

for all $n\in\mathbb{N}\cup\{0\}$. From Equation (45), the following equations are obtained:

$$\zeta_0-S(\varrho_n)=(1-\mu)(\zeta_0-\varrho_n)+\mu(\zeta_0-\Psi(\varrho_n))\ , \tag{51}$$

and

$$\zeta_0-\Psi(\varrho_n)=\frac{1}{\mu}(\zeta_0-S(\varrho_n))-\frac{(1-\mu)}{\mu}(\zeta_0-\varrho_n). \tag{52}$$

Now:

$$\begin{aligned}\|\zeta_0-\varrho_{n+1}\|&=\|\zeta_0-\Psi_\alpha(\varrho_n)\|\\ &=\|(1-\alpha)(\zeta_0-\varrho_n)+\alpha(\zeta_0-\Psi(\varrho_n))\|.\end{aligned} \tag{53}$$

From Equation (52), the following equation is obtained:

$$\begin{aligned}\|\zeta_0-\varrho_{n+1}\|&=\left\|(1-\alpha)(\zeta_0-\varrho_n)+\frac{\alpha}{\mu}(\zeta_0-S(\varrho_n))-\frac{\alpha(1-\mu)}{\mu}(\zeta_0-\varrho_n)\right\|\\ &=\left\|\left(1-\frac{\alpha}{\mu}\right)(\zeta_0-\varrho_n)+\frac{\alpha}{\mu}(\zeta_0-S(\varrho_n))\right\|.\end{aligned} \tag{54}$$

Take $\beta = \frac{\alpha}{\mu}$, then $\beta \in (0,1)$. From Equation (50), the following equation is obtained:

$$\|\zeta_0 - \varrho_{n+1}\| = \|(1-\beta)(\zeta_0 - \varrho_n) + \beta(\zeta_0 - S(\varrho_n))\|, \quad (55)$$

$$\leq (1-\beta)\|\zeta_0 - \varrho_n\| + \beta\|\zeta_0 - S(\varrho_n)\| = \|\zeta_0 - \varrho_n\|. \quad (56)$$

Therefore, the sequence $\{\|\zeta_0 - \varrho_n\|\}$ is bounded by $u_0 = \|\zeta_0 - \varrho_0\|$. If $\varrho_{n_0} = \zeta_0$ for any $n_0 \in \mathbb{N}$ then from Equation (56), $\varrho_n \to \zeta_0$ as $n \to \infty$. If $\varrho_n \neq \zeta_0$ for all $n \in \mathbb{N}$, take the following conditions:

$$z_n = \frac{\zeta_0 - \varrho_n}{\|\zeta_0 - \varrho_n\|} \text{ and } z'_n = \frac{\zeta_0 - S(\varrho_n)}{\|\zeta_0 - \varrho_n\|}. \quad (57)$$

If $\beta \leq \frac{1}{2}$ and from Equation (55), we have the following equation:

$$\begin{aligned}
\|\zeta_0 - \varrho_{n+1}\| &= \|(1-\beta)(\zeta_0 - \varrho_n) + \beta(\zeta_0 - S(\varrho_n))\| \\
&= \|\zeta_0 - (1-\beta)\varrho_n - \beta S(\varrho_n)\| \\
&= \|\zeta_0 - \varrho_n + \beta\varrho_n - \beta S(\varrho_n) - 2\beta\zeta_0 + 2\beta\zeta_0 + \beta\varrho_n - \beta\varrho_n\| \\
&= \|\zeta_0(1-2\beta) - \varrho_n(1-2\beta) + (2\beta\zeta_0 - \beta\varrho_n - \beta S(\varrho_n))\| \\
&\leq (1-2\beta)\|\zeta_0 - \varrho_n\| + \beta\|2\zeta_0 - \varrho_n - S(\varrho_n)\| \\
&= 2\beta\|\zeta_0 - \varrho_n\| \frac{\|z_n + z'_n\|}{2} + (1-2\beta)\|\zeta_0 - \varrho_n\|.
\end{aligned} \quad (58)$$

Since $\mathcal{X}$ is a uniformly convex Banach space, by using the definition of uniformly convex space with $\|z_n\| \leq 1$, $\|z'_n\| \leq 1$, and the following equation:

$$\|z_n - z'_n\| = \frac{\|\varrho_n - S\varrho_n\|}{\|\zeta_0 - \varrho_n\|} \geq \frac{\|\varrho_n - S\varrho_n\|}{u_0} = \varepsilon \text{ (say)}. \quad (59)$$

Noting that modulus of convexity $\delta(\varepsilon)$ is a nondecreasing function of ε, we obtain the following equation:

$$\frac{\|z_n + z'_n\|}{2} \leq 1 - \delta\left(\frac{\|\varrho_n - S(\varrho_n)\|}{u_0}\right). \quad (60)$$

From Equations (60) and (62), the following equation is obtained:

$$\begin{aligned}
\|\zeta_0 - \varrho_{n+1}\| &\leq \left(2\beta\left(1 - \delta\left(\frac{\|\varrho_n - S(\varrho_n)\|}{u_0}\right)\right) + (1-2\beta)\right)\|\zeta_0 - \varrho_n\| \\
&= \left(1 - 2\beta\delta\left(\frac{\|\varrho_n - S(\varrho_n)\|}{u_0}\right)\right)\|\zeta_0 - \varrho_n\|.
\end{aligned} \quad (61)$$

Using induction in the above inequality, it follows that:

$$\|\zeta_0 - \varrho_{n+1}\| \leq \prod_{j=0}^{n}\left(1 - 2\beta\delta\left(\frac{\|\varrho_j - S(\varrho_j)\|}{u_0}\right)\right)u_0. \quad (62)$$

We shall prove that $\lim_{n\to\infty}\|S(\varrho_n) - \varrho_n\| = 0$. Arguing by contradiction, consider that $\|S(\varrho_n) - \varrho_n\|$ does not converge to zero. Then, there exists a subsequence $\{\varrho_{n_k}\}$ of $\{\varrho_n\}$ such that $\|S(\varrho_{n_k}) - \varrho_{n_k}\|$ converges to $\eta > 0$. Since $\delta(\cdot) \in [0,1]$ is nondecreasing and $\beta \leq \frac{1}{2}$, we have $1 - 2\beta\delta(\frac{\|\varrho_k - S(\varrho_k)\|}{u_0}) \in [0,1]$ for all $k \in \mathbb{N}$. From Equation (63) and for sufficiently large k, we have the following equation:

$$\|\zeta_0 - \varrho_{n_{k+1}}\| \leq \left(1 - 2\beta\delta\left(\frac{\eta}{2u_0}\right)\right)^{(n_{k+1})}u_0. \quad (63)$$

Making $k \to \infty$, it follows that $\varrho_{n_k} \to \zeta_0$. By Equation (46), we get $S(\varrho_{n_k}) \to \zeta_0$ and $\|\varrho_{n_k} - S(\varrho_{n_k})\| \to 0$ as $k \to \infty$, which is a contradiction.

If $\beta > \frac{1}{2}$ then $1 - \beta < \frac{1}{2}$ because $\beta \in (0,1)$. Now:

$$\begin{aligned}\|\zeta_0-\varrho_{n+1}\| &= \|\zeta_0-(1-\beta)\varrho_n-\beta S(\varrho_n)\| \\ &= \|\zeta_0-\varrho_n+\beta\varrho_n-\beta S(\varrho_n)+(2-2\beta)\zeta_0-(2-2\beta)\zeta_0 \\ &\quad +S(\varrho_n)-S(\varrho_n)+\beta S(\varrho_n)-\beta S(\varrho_n)\| \\ &= \|(2\zeta_0-\varrho_n-S(\varrho_n))-\beta(2\zeta_0-\varrho_n-S(\varrho_n))+2\beta(\zeta_0-S(\varrho_n)), \\ &\quad -(\zeta_0-S(\varrho_n))\| \\ &\le (1-\beta)\|2\zeta_0-\varrho_n-S(\varrho_n)\|+(2\beta-1)\|\zeta_0-\varrho_n\| \\ &\le 2(1-\beta)\|\zeta_0-\varrho_n\|\frac{\|z_n+z_n'\|}{2}+(2\beta-1)\|\zeta_0-\varrho_n\|\end{aligned} \tag{64}$$

and by the uniform convexity of $\mathscr{X}$, we get the following equation:

$$\|\zeta_0-\varrho_{n+1}\| \le \left(2(1-\beta)-2(1-\beta)\delta\left(\frac{\|\varrho_n-S(\varrho_n)\|}{u_0}\right) + (2\beta-1)\right)\|\zeta_0-\varrho_n\|. \tag{65}$$

Using induction in the above inequality, we get the following equation:

$$\|\zeta_0-\varrho_{n+1}\| \le \prod_{j=0}^{n}\left(1-2(1-\beta)\delta\left(\frac{\|\varrho_j-S(\varrho_j)\|}{u_0}\right)\right)u_0. \tag{66}$$

Using the similar argument as in the previous case, it can be easily shown that $\|S(\varrho_n)-\varrho_n\|\to 0$ as $n\to\infty$. Therefore, in both cases, $\|S(\varrho_n)-\varrho_n\|\to 0$ as $n\to\infty$. From Equation (47), $\|\Psi(\varrho_n)-\varrho_n\|\to 0$ as $n\to\infty$, Ψ_α is asymptotically regular and this completes the proof. □

Remark 2. The above theorem is a generalization of [18, Theorem 1] for a more general class of mappings.

Theorem 9. *Let $\mathscr{B}$ be a nonempty closed subset of a Banach space $\mathscr{X}$ and $S:\mathscr{B}\to\mathscr{B}$ a quasinonexpansive mapping. Assume that $\mathscr{X}$ is strictly convex and $\mathscr{B}$ is a convex compact subset of $\mathscr{X}$. If S is continuous then for any $\zeta_0\in\mathscr{B}, \alpha\in(0,1)$, the α-Krasnosel'skiĭ process $\{S_\alpha^n(\zeta_0)\}$ converges to some $\zeta^*\in F(S)$.*

Proof. Following the same line of proof of [7, Theorem 5], one can complete the proof. □

Theorem 10. *Let $\mathscr{B}$ and $\mathscr{X}$ be the same as in Theorem 6. Let $\Psi:\mathscr{B}\to\mathscr{B}$ be a b-enriched quasinonexpansive mapping with $F(\Psi)\neq\emptyset$. If Ψ is continuous then for any $\zeta_0\in\mathscr{B}, \alpha\in(0,\frac{1}{b+1})$, the α-Krasnosel'skiĭ process $\{\Psi_\alpha^n(\zeta_0)\}$ converges to some $\zeta^*\in F(\Psi)$.*

Proof. Following the same proof technique as in Theorem 4, we can define a mapping $S:\mathscr{B}\to\mathscr{B}$ as follows:

$$S(\zeta)=\left(1-\frac{1}{b+1}\right)\zeta+\frac{1}{b+1}\Psi(\zeta) \text{ for all } \zeta\in\mathscr{B}, \tag{67}$$

and S is a quasinonexpansive mapping. For given $\zeta_0\in\mathscr{B}$, $\beta\in(0,1)$, we can define a sequence $\{\zeta_n\}$ as follows:

$$\zeta_n=S_\beta^n(\zeta_1)=(1-\beta)\zeta_{n-1}+\beta S(\zeta_{n-1}). \tag{68}$$

From Theorem 6, the sequence $\{\zeta_n\}$ converges to some $\zeta^*\in F(S)$. Using the definition of S, we have the following equation:

$$\zeta_n=(1-\beta)\zeta_{n-1}+\beta S(\zeta_{n-1})=\left(1-\beta\frac{1}{b+1}\right)\zeta_{n-1} +\beta\frac{1}{b+1}\Psi(\zeta_{n-1}). \tag{69}$$

Take $\alpha=\frac{\beta}{b+1}\in[0,\frac{1}{b+1})$, then, we obtain the following equation:

$$\zeta_n=(1-\alpha)\zeta_{n-1}+\alpha\Psi(\zeta_{n-1}). \tag{70}$$

Hence, $\{\zeta_n\}=\{\Psi_\alpha^n(\zeta_0)\}$ strongly converges to a fixed point of S. But $F(S)=F(\Psi)$. This completes the proof. □

Theorem 11. *Let $\mathscr{B}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$. Let $S:\mathscr{B}\to\mathscr{B}$ be a quasinonexpansive mapping with $F(S)\neq\emptyset$ and P the metric projection from $\mathscr{X}$ into $F(S)$. Then for each $\zeta\in\mathscr{B}$, the sequence $\{PS^n(\zeta)\}$ converges to some $\varrho\in F(S)$.*

Proof. Following the same line of proof of [7, Theorem 6], one can complete the proof. □

Theorem 12. *Let $\mathscr{B}, \mathscr{X}$, and P be same as in Theorem 8. Let $\Psi:\mathscr{B}\to\mathscr{B}$ be a b-enriched quasinonexpansive mapping with $F(\Psi)\neq\emptyset$. Then for each $\zeta\in\mathscr{B}$, $\alpha=\frac{1}{b+1}$, the α-Krasnosel'skiĭ process $\{\Psi_\alpha^n(\zeta)\}$ converges to some $\varrho\in F(\Psi)$.*

Proof. In view of Theorems 4 and 8, one can complete the proof. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Open access funding was enabled and organized by SANLiC Gold.

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