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Numerical Solution of Nonlinear Equations by a Twelth-Order Iterative Method with Memory

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Author's contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Some real life mathematical problems can be converted in the form of nonlinear equations. Solving such problems by analytical approaches is difficult in many situations. Hence numerical solution is the best way in this case. In this paper, a twelth-order iterative scheme for solving nonlinear equations is presented and analyzed in terms of efficiency. The new scheme is derived from the well-known King's method with order of convergence eight. We extend eighth-order King's method to an iterative method with memory of order 12.16 by using famous Newton's interpolating polynomial of degree 6 to avoid the derivative used in King's method. The new derived method is a three-step and is totally derivative free with twelth order of convergence. The method requires four functional evaluations at each iteration introducing high efficiency index of $(12.16)^{\frac{1}{4}} = 1.8673$. Convergence order of new method is also studied. It is achieved by using matrix method of Herzburger. Numerical results are also provided to support theoretical analysis. Comparison of the derived scheme with previously well-known iterative schemes of the same order is also presented. As different schemes of same order has efficiency index of $(12)^{\frac{1}{6}} = 1.5131$ because they requires six functional evaluations at each iteration, hence the proposed scheme is better than other schemes.

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1 Introduction

It is well known that a broad class of problems that appear in many fields of pure and applied research may be explored in the general context of nonlinear equations. Due to their significance, a number of numerical approaches for solving nonlinear equations have been proposed and examined under certain circumstances. The construction of these numerical methods used a variety of methodologies, including the Taylor series, the homotopy perturbation method and its variants, the quadrature formula, the variational iteration method, and the decomposition method. Solving nonlinear equation $f(x) = 0$ using iterative schemes is a classical problem in field of numerical computation [1]-[6] and references therein. Iterative schemes also has a great importance in Nanotechnology, in which solution of fractional differential-difference equations [7] is obtained using these schemes. In [8], Nasir ali et al. proposed a new iterative scheme for solving important nonlinear equations in field of fractional calculus.

Jarratt and Ostrowski presented somewell-known two-point techniques. King [9] presented one of the most famous optimal 4^{th}order iterative method. But flaws of this scheme is that it requires first derivative in each step. Many authors modified King's method to get more accurate results such as Chun [10] introduced King's like methods of order four, but computing the first derivative within the iteration is also needed.

Chun and Lee presented a new 4^{th} order optimal root-finding method to solve non-linear equations which describe the conjugacy classes and dynamics of the presented optimal method for complex polynomials of degree two and three. They obtained Jarratt's scheme of fourth order as a special case. Behl [2]-[4] introduced a fourth-order derivative-free scheme which is a modification in King's method by using weight functions.

To achieve more accuracy with less computation many authors introduced Optimal methods of eighth-order of convergence i.e Chun et al. [10] Cordero et al. [11]-[12], Behl et al.[2]-[4] and Geum et al. [13]. Geum and Neta [13] developed a 16^{th} order simple root finding optimal method with general weight functions.

Our aim is to present a derivative-free iterative method with memory of twelth order using King's iterative schemes and steffensen approaches. The new scheme is derived from the well-known King's method with order of convergence eight. We extend eighth-order King's method to an iterative method with memory of order 12.16 by using famous Newton's interpolating polynomial of degree 6 to avoid the derivative used in King's method. The new derived method is a three-step and is totally derivative free with twelth order of convergence. This paper is organized in the following manner.

In the next section we establish our new method and gave convergence analysis. In section 3, solution of some numerical examples with their comparison to other well-known iterative schemes are presented. Section 4 is a smart conclusion.

2 Establishment of new scheme of twelth-order

We start with king's [9] technique that is one of important family for finding solution of nonlinear problems.

$$y_m = x_m - \frac{f(x_m)}{f'(x_m)},$$
$$x_{m+1} = y_m - \frac{f(y_m)}{f'(x_m)}.\frac{f(x_m)+\gamma f(y_m)}{f(x_m)+(\gamma-2)f(y_m)},\ \ (m=0,1......),\ \gamma\in R,$$

Here x_0 is a preliminary approximation of a simple zero α of f. First we derive an optimal derivative free scheme of 2 point method having convergence order 4. We consider Steffensen's scheme for the 1^{st} & 2^{nd} step approximation $f'(x_m)$

$$f'(x_n) \approx \frac{f[y_m, w_m]}{G(t_m)},$$

while $w_m = x_m - \beta f(x_m)$, $\beta \neq 0$, $f[y_m, w_m] = \frac{f(y_m)-f(w_m)}{y_m - w_m}$, $t_m = \frac{f(y_m)}{f(x_m)}$. Here G is actual function. Hence, we obtain a fourth order scheme,

$$\begin{cases} y_m = x_m - \frac{\beta f(x_m)^2}{f(x_m)-f(w_m)}, \\ x_{m+1} = y_m - \frac{f(x_m)+\gamma f(y_m)}{f(x_m)+(\gamma-2)f(y_m)}.\frac{f(y_m)}{f[y_m,w_m]}G(t_m). \end{cases} \tag{1}$$

Now we derive eighth-order scheme by adding Newton step in scheme (1), we have

$$\begin{cases} y_m = x_m - \frac{\beta f(x_m)^2}{f(x_m)-f(w_m)}, \\ z_m = y_m - \frac{f(x_m)+\gamma f(y_m)}{f(x_m)+(\gamma-2)f(y_m)}.\frac{f(y_m)}{f[y_m,w_m]}G(t_m). \\ x_{m+1} = z_m - \frac{f(z_m)}{f'(z_m)}. \end{cases} \tag{2}$$

It is observed that function is evaluated many times to make it derivative-free and optimal method. We approximate $f'(z_m)$ with Newton's interpolation of degree 3 at the point x_m, y_m, and z_m.

$$\begin{aligned} N_3(t; z_m, y_m, x_m, w_m) = {} & f(z_m) + f[z_m, y_m](t - z_m) \\ & + f[z_m, y_m, x_m](t - z_m)(t - y_m) + f[z_m, y_m, x_m, w_m] \\ & (t - z_m)(t - y_m)(t - x_m). \end{aligned}$$

It can be seen

$$N_3(z_m) = f(z_m),\ \text{and}\ N_3'(t)\,|_{t=z_m} = f'(z_m).$$

So,

$$\begin{aligned} N_3'(z_m) &= \left[\frac{d}{dt}N_3(t)\right]_{t=z_m} \\ &= f[z_m, y_m] + f[z_m, y_m, x_m](z_m - y_m) + f[z_m, y_m, x_m, w_m] \\ &\quad (z_m - y_m)(z_m - x_m). \end{aligned}$$

hence we get

$$\begin{cases} y_m = x_m - \frac{\beta f(x_m)^2}{f(x_m)-f(w_m)}, \\ z_m = y_m - \frac{f(x_m)+\gamma f(y_m)}{f(x_m)+(\gamma-2)f(y_m)}.\frac{f(y_m)}{f[y_m,w_m]}G(t_m). \\ x_{m+1} = z_m - \frac{f(z_m)}{f[z_m,y_m]+f[z_m,y_m,x_m](z_m-y_m)+f[z_m,y_m,x_m,w_m](z_m-y_m)(z_m-x_m)}. \end{cases} \tag{3}$$

which is iterative scheme with memory of order 8. Now we extend this method to achieve convergence order 12.

This is done by using the speed acceleration parameters in scheme (3). If $\beta \neq 1/f'(\alpha)$ rate of convergence of scheme (3) is 8. When $\beta = 1/f'(\alpha)$ rate of convergence of method (3) could be twelve. Because the value of $f'(\alpha)$ is unavailable, we apply an approximation $f'(\alpha) \approx f'(x_m)$. Our objective is to create a with memory method that includes parameter calculations $\beta = \beta_m$ as iteration progresses by $\beta_m = 1/f'(\alpha)$ for $m = 1, 2, 3....$ Initial value β_0 should be selected before begning of iteration process. Here we use some characters $\rightarrow, O$ and $\sim$ according to Traub's iterative sccheme.

If $\lim_{m\to\infty} f(x_m) = C$, we write down $f(x_m) \rightarrow C$ or $f \rightarrow C$, where C is a non zero contants. If $_m \rightarrow C$, we write $f = O(g)$ or $f \sim C(g)$.

By approximating $f'(\alpha)$ with $N_4'(x_m)$ we get

$$\beta_m = \frac{1}{N_4'(x_m)},$$

Here $N_4^{\acute{}}(t_m) := N_4(t; x_m, z_{m-1}, y_{m-1}, w_{m-1}, x_{m-1})$ is Newton's interpolation polynomial of 4^{th} degree, place with 5 approximation $(x_m, z_{m-1}, y_{m-1}, w_{m-1}, x_{m-1})$.

$$\begin{aligned} N_4'(x_m) = \left[\frac{d}{dt} N_4(t)\right]_{t=x_m} &= f[x_m, z_{m-1}] + f[x_m, z_{m-1}, y_{m-1}](x_m - z_{m-1}) \\ &+ f[x_m, z_{m-1}, y_{m-1}, w_{m-1}](x_m - z_{m-1})(x_m - y_{m-1}) \\ &+ f[x_m, z_{m-1}, y_{m-1}, w_{m-1}, x_{m-1}](x_m, z_{m-1})(x_m - y_{m-1}) \\ &(x_m - w_{m-1}). \end{aligned}$$

We approximate in (2) $f'(z_m)$ with Newton's interpolation of degree 6 at the point $z_m, y_m, w_m, x_m, z_{m-1}$ and y_{m-1}.

$$\begin{aligned} &N_5(t; z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}) = f(z_m) + f[z_m, y_m](t - z_m) + f[z_m, y_m, w_m] \\ &(t - z_m)(t - y_m) + f[z_m, y_m, w_m, x_m](t - z_m) \\ &(t - y_m)(t - w_m) + f[z_m, y_m, w_m, x_m, z_{m-1}] \\ &(t - z_m)(t - y_m)(t - w_m)(t - x_m) + \\ &f[z_m, y_m, w_m, x_m, z_{m-1}, , y_{m-1}](t - z_m) \\ &(t - y_m)(t - w_m)(t - x_m)(t - z_{m-1}). \end{aligned}$$

It is clear that,
$N_5(z_m) = f(z_m)$, *and* $N_5'(t)\,|_{t=z_m} = f'(z_m)$.
Then,

$$\begin{aligned} N_5'(z_m) = \left[\frac{d}{dt} N_5(t)\right]_{t=z_m} &= f[z_m, y_m] + f[z_m, y_m, w_m](z_m - y_m) + f[z_m, y_m, w_m, x_m] \\ &(z_m - y_m)(z_m - w_m) + f[z_m, y_m, w_m, x_m, z_{m-1}](z_m - y_m)(z_m - w_m)(z_m - x_m) \\ &+ f[z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}](z_m - y_m)(z_m - w_m)(z_m - x_m)(z_m - z_{m-1}). \end{aligned}$$

and hence we get,

$$\begin{aligned} y_m &= x_m - \frac{\beta_m f(x_m)^2}{f(x_m)-f(w_m)}, \\ z_m &= y_m - \frac{f(x_m)+\gamma f(y_m)}{f(x_m)+(\gamma-2)f(y_m)} \cdot \frac{f(y_m)}{f[y_m,w_m]} G(t_m), \\ x_{m+1} &= z_m - \frac{f(z_m)}{f[z_m,y_m]+f[z_m,y_m,w_m](z_m-y_m)+...+f\left[z_m,y_m,w_m,x_m,z_{m-1},y_{m-1}\right](z_m-y_m)...\left(z_m-w_{m-1}\right)}. \end{aligned} \tag{4}$$

We denote the above scheme with AM12 which is a with memory method of convergence order 12.16. Now, we will prove the convergence order by applying the matrix method of Herzberger.

Theorem 2.1. *Let x_0 be a starting value which is close enough to 0 of $f(x)$ and the iterative scheme AM12 has 2 parameters which are repeatedly computed by the outline, then the scheme AM12 has 12.164414 order of convergence.*

Proof. By using Herzberger's matrix method we will find R-order of convergence which states that the spectral radius of matrix $M^{(u)} = (t_{p;q})(1 \leq p; q \leq u)$ related to a with-memory 1 step r-point scheme $x_k = \Phi\left(x_{k-1}, x_{k-2}, ..., x_{k-u}\right)$ is the lower bound of its rate of convergence. The elements of this method are as:

$t_{p,q} =$ no.of functional evaluations needed at point $\mathrm{x}_{k-q} = 1, 2, ..., u$

$t_{p,q-1} = 1$ for $p = 2, 3, ..., u$

$t_{p,q} = 0$, otherwise.

Moreover, the spectral radius of product of the matrices $B_1, B_2, ...B_m$ is the lower bound of order of an $m-$step method $\Phi = \Phi_1, \Phi_2, ..., \Phi_m$ where the matrices B_k correspond to the iteration step $\Phi_{k,} 1 \leq K \leq m$. From the above equations we develop the associated matrices as follow:

$$X_{k+1} = \Phi_1\left(z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}\right).$$

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$Z_m = \Phi_2\left(y_m, w_m, x_m, z_{m-1}, y_{m-1}, w_{m-1}\right).$$

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$Y_m = \Phi_3\left(w_m, x_m, z_{m-1}, y_{m-1}, w_{m-1}, x_{m-1}\right).$$

$$M_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_m = \Phi_4\left(x_m, z_{m-1}, y_{m-1}, w_{m-1}, x_{m-1}, z_{m-2}\right).$$

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_1 \cdot M_2 = \begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_1 \cdot M_2 \cdot M_3 = \begin{bmatrix} 4 & 4 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence we obtained,

$$\begin{aligned} M^{(4)} &= M_1 \cdot M_2 \cdot M_3 \cdot M_4 \\ &= \begin{bmatrix} 8 & 4 & 4 & 4 & 4 & 4 \\ 4 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The eigen values of $M^{(4)}$ are

$$\begin{aligned} \lambda_1 &= 12.164414 \\ \lambda_2 &= -0.164414003 \\ \lambda_3 &= 2.0470017e^{-15} - 3.78075673e^{-81} \\ \lambda_4 &= 2.75578744e^{-15} - 3.78075673e^{-81} \\ \lambda_5 &= 1.22388505e^{-16} + 1.13323081e^{-16} \end{aligned}$$

Hence spectral radius of $M^{(4)}$matrix is 12.164414 which is convergence order of the method. □

3 Numerical Examples and comparison

3.1 Method 1

Consider G as weight function

$$G(t_m) = 1 - t_m, \qquad (5)$$

where $t_m = \frac{f(y_m)}{f(x_m)}$. Hence we get the following scheme denoted by AM.1.

$$
\begin{aligned}
y_m &= x_m - \frac{\beta_m f(x_m)^2}{f(x_m) - f(w_m)}, w_m = x_m - \beta_m f(x_m), \beta_m = \frac{1}{N_5'(x_m)}, \\
z_m &= y_m - \frac{f(x_m) + \gamma f(y_m)}{f(x_m) + (\gamma - 2) f(y_m)} . \frac{f(x_m) - f(y_m)}{f(x_m)} . \frac{f(y_m)}{f[y_m, w_m]} \\
x_{m+1} &= z_m - \left[\frac{f(z_m)}{\begin{array}{c} f[z_m, y_m] + f[z_m, y_m, w_m](z_m - y_m) + .. \\ . + f[z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}](z_m - y_m) \ldots (z_m - w_{m-1}) \end{array}} \right]
\end{aligned}
$$

3.2 Method 2

Consider G as a weight function

$$G(t_m) = 1 - \frac{t_m}{1 + t_m},$$

where $t_m = \frac{f(y_m)}{f(x_m)}$.

$$
\begin{aligned}
y_m &= x_m - \frac{\beta_m f(x_m)^2}{f(x_m) - f(w_m)}, w_m = x_m - \beta_m f(x_m), \beta_m = \frac{1}{N_5'(x_m)} \\
z_m &= y_m - \frac{f(x_m) + \gamma f(y_m)}{f(x_m) + (\gamma - 2) f(y_m)} . \frac{f(x_m)}{f(x_m) + f(y_m)} . \frac{f(y_m)}{f[y_m, w_m]}, \\
x_{m+1} &= z_m - \frac{f(z_m)}{f[z_m, y_m] + f[z_m, y_m, w_m](z_m - y_m) + \ldots + f\left[z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}\right](z_m - y_m) \ldots \left(z_m - w_{m-1}\right)}.
\end{aligned}
$$

we call this method AM.2.

3.3 Method 3.

Choose weight function G

$$G(t_m) = \frac{1 - 2t_m}{1 - t_m},$$

where $t_m = \frac{f(y_m)}{f(x_m)}$

$$
\begin{aligned}
y_m &= x_m - \frac{\beta_m f(x_m)^2}{f(x_m) - f(w_m)}, w_m = x_m - \beta_m f(x_m), \beta_m = \frac{1}{N_5'(x_m)} \\
z_m &= y_m - \frac{f(x_m) + \gamma f(y_m)}{f(x_m) + (\gamma - 2) f(y_m)} . \frac{f(x_m) - 2f(y_m)}{f(x_m) - f(y_m)} . \frac{f(y_m)}{f[y_m, w_m]}, \\
x_{m+1} &= z_m - \frac{f(z_m)}{f[z_m, y_m] + f[z_m, y_m, w_m](z_m - y_m) + \ldots + f\left[z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}\right](z_m - y_m) \ldots \left(z_m - w_{m-1}\right)}.
\end{aligned}
$$

we call this scheme AM.3.

3.4 Method 4.

Consider G as weight function

$$G(t_m) = (1 - t_m)^{\frac{2t_m + 1}{t_m + 1}},$$

$$
\begin{aligned}
y_m &= x_m - \frac{\beta_m f(x_m)^2}{f(x_m) - f(w_m)}, w_m = x_m - \beta_m f(x_m), \beta_m = \frac{1}{N_5'(x_m)} \\
z_m &= y_m - \frac{f(x_m) + \gamma f(y_m)}{f(x_m) + (\gamma - 2) f(y_m)} . \left(\frac{f(x_m) - f(y_m)}{f(x_m)} \right)^{\frac{2f(y_m) + f(x_m)}{f(y_m) + f(x_m)}} . \frac{f(y_m)}{f[y_m, w_m]}, \\
x_{m+1} &= z_m - \frac{f(z_m)}{f[z_m, y_m] + f[z_m, y_m, w_m](z_m - y_m) + \ldots + f\left[z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}\right](z_m - y_m) \ldots \left(z_m - w_{m-1}\right)}.
\end{aligned}
$$

this scheme is named as AM.4.

3.5 Kung and Traub(KT)

The derivative free method by Kung and Traub[14],

$$\begin{gathered} y_m = x_m - \frac{f(x_m)}{f[x_m, w_m]}, w_m = x_m + \beta_m f(x_m), \beta_m = \frac{1}{N'(x_m)} \\ z_m = y_m - \frac{f(y_m) f(w_m)}{(f(w_m) - f(y_m)) f[x_m, w_m]}, \\ x_{m+1} = z_m - \frac{f(y_m) f(w_m) \left(y_m - x_m \frac{f(x_m)}{f[x_m, z_m]} \right)}{(f(y_m) - f(z_m))(f(w_m) - f(z_m))} + \frac{f(y_m)}{f[y_m, z_m]}. \end{gathered}$$

this method is named as KT.

3.6 Sharma et al.

The method by Sharma et al. [15]

$$\begin{gathered} y_m = x_m - \frac{f(x_m)}{\varphi(x_m)}, \varphi(x_m) = \frac{f(w_m) - f(x_m)}{\beta_m f(x_m)}, w_m = x_m + \beta_m f(x_m), \beta_m = \frac{-1}{N'(x_m)} \\ z_m = y_m - H(\mu_m \upsilon_m) \frac{f(y_m)}{\varphi(x_m)}, H(\mu_m \upsilon_m) = \frac{1+\mu_m}{1-\upsilon_m}, \upsilon_m = \frac{f(y_m)}{f(w_m)}, \mu_m = \frac{f(y_m)}{f(x_m)} \\ x_{m+1} = z_m - \frac{f(z_m)}{f[z_m, y_m] + f[z_m, y_m, x_m](z_m - y_m) + f[z_m, y_m, x_m, w_m](z_m - y_m)(z_m - x_m)}. \end{gathered}$$

3.7 Zheng et al.

The method by Zheng et al. [16]

$$\begin{gathered} y_m = x_m - \frac{f(x_m)}{f[x_m, w_m]}, w_m = x_m + \beta_m f(x_m), \beta_m = \frac{-1}{N'(x_m)} \\ z_m = y_m - \frac{f(y_m)}{f[y_m, x_m] + f[y_m, x_m, w_m](y_m - x_m)}, \\ x_{m+1} = z_m - \frac{f(z_m)}{f[z_m, y_m] + f[z_m, y_m, x_m](z_m - y_m) + f[z_m, y_m, x_m, w_m](z_m - y_m)(z_m - x_m)}. \end{gathered}$$

In order to test our presented with-memory method, we select the following nonlinear functions with initial approximation as x_0 and exact solution α. The comparison is based on computational order of convergence and error computation.

Table 1. Test functions, exact root α and initial approximation x_0

Test Functions	α	x_0
$f_1(x) = \ln\left(x^2 - 2x + 2\right) + e^{x^2 - 5x + 4} \sin(x - 1)$	1	1.05
$f_2(x) = e^{x^2 + x\cos(x) - 1} \sin(\pi x) + x \ln(x \sin(x) + 1)$	0	0.2
$f_3(x) = \frac{\left(1 - \sin\left(x^2\right)\right)\left(1 + x^2\right)}{1 + x^3} + x \ln\left(x^2 - \pi + 1\right) - \frac{1+\pi}{1+\pi^{3/2}}$	1.77	1.74
$f_4(x) = e^{x^2 - 3x} \sin(x) + \ln\left(x^2 + 1\right)$	0	0.35
$f_5(x) = \left(4 + 3\sin(x) - 2x^2\right)^4$	1.854	1.86

Table 2. Errors and coc of AM12

f_n	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	COC
f_1	5.49×10^{-8}	1.65×10^{-84}	9.27×10^{-1003}	12.9997
f_2	4.13×10^{-6}	9.44×10^{-66}	1.93×10^{-781}	12.9997
f_3	9.97×10^{-12}	5.28×10^{-134}	2.72×10^{-1601}	12.9997
f_4	3.42×10^{-6}	5.63×10^{-63}	2.24×10^{-744}	12.9997
f_5	2.52×10^{-3}	9.61×10^{-4}	3.61×10^{-4}	12.9997

$x_0 = 1.05$

Table 3. Error and coc of methods at f_1

	Scheme AM.1	Scheme AM.2	Scheme AM.3	Scheme AM.4
$\mid x_1 - \alpha \mid$	0.314×10^{-6}	0.314×10^{-5}	0.543×10^{-6}	5.49×10^{-8}
$\mid x_2 - \alpha \mid$	0.110×10^{-66}	0.153×10^{-61}	0.715×10^{-65}	1.65×10^{-84}
$\mid x_3 - \alpha \mid$	0.178×10^{-800}	0.914×10^{-739}	0.100×10^{-788}	9.27×10^{-1003}
COC	12.1378	12.1056	12.1238	12.9997

$x_0 = 0.2$

Table 4. Error and coc of methods f_2

	Scheme AM.1	Scheme AM.2	Scheme AM.3	Scheme AM.4
$\mid x_1 - \alpha \mid$	0.900×10^{-4}	0.585×10^{-3}	0.308×10^{-4}	4.13×10^{-6}
$\mid x_2 - \alpha \mid$	0.111×10^{-44}	0.713×10^{-38}	0.650×10^{-49}	9.44×10^{-66}
$\mid x_3 - \alpha \mid$	0.255×10^{-536}	0.126×10^{-454}	0.419×10^{-587}	1.93×10^{-781}
COC	12.0178	11.9363	12.0465	12.9997

$x_0 = 1.74$

Table 5. Error and coc of methods at f_3

	Scheme AM.1	**Scheme AM.2**	**Scheme AM.3**	**Scheme AM.4**
$\mid x_1 - \alpha \mid$	0.836×10^{-8}	0.164×10^{-7}	0.605×10^{-8}	9.97×10^{-12}
$\mid x_2 - \alpha \mid$	0.102×10^{-94}	0.101×10^{-96}	0.624×10^{-97}	5.28×10^{-134}
$\mid x_3 - \alpha \mid$	0.134×10^{-1138}	0.118×10^{-1090}	0.341×10^{-1165}	2.72×10^{-1601}
COC	12.0110	12.0173	12.0048	12.9997

$x_0 = 0.35$

Table 6. Error and coc of methods at f_4

	Scheme AM.1	**Scheme AM.2**	**Scheme AM.3**	**Scheme AM.4**
$\mid x_1 - \alpha \mid$	0.451×10^{-6}	0.290×10^{-7}	0.539×10^{-8}	3.42×10^{-12}
$\mid x_2 - \alpha \mid$	0.612×10^{-63}	0.716×10^{-94}	0.335×10^{-96}	5.63×10^{-135}
$\mid x_3 - \alpha \mid$	0.354×10^{-744}	0.612×10^{-1090}	0.148×10^{-1138}	2.24×10^{-1475}
COC	12.0110	12.0173	12.0048	12.9997

$x_0 = 1.86$

Table 7. Error and coc of methods at f_5

	Scheme AM.1	**Scheme AM.2**	**Scheme AM.3**	**Scheme AM.4**
$\mid x_1 - \alpha \mid$	$\mathbf{4.12}\times 10^{-3}$	$\mathbf{3.21}\times 10^{-5}$	$\mathbf{2.23}\times 10^{-8}$	2.52×10^{-14}
$\mid x_2 - \alpha \mid$	$\mathbf{0.542}\times 10^{-94}$	$\mathbf{0.453}\times 10^{-96}$	$\mathbf{0.342}\times 10^{-45}$	9.61×10^{-165}
$\mid x_3 - \alpha \mid$	$\mathbf{0.341}\times 10^{-1138}$	$\mathbf{0.614}\times 10^{-1090}$	$\mathbf{0.360}\times 10^{-999}$	3.61×10^{-1340}
COC	12.0110	12.0173	12.0048	12.9997

$f_1, x_0 = 1.35$

Table 8. Comparison of different methods

	KT	Sharma	Zheng
$\|x_1 - \alpha\|$	0.845×10^{-4}	0.308×10^{-6}	0.148×10^{-5}
$\|x_2 - \alpha\|$	0.393×10^{-45}	0.179×10^{-67}	0.157×10^{-61}
$\|x_3 - \alpha\|$	0.100×10^{-540}	0.126×10^{-812}	0.481×10^{-738}
COC	11.9906	12.1688	12.0973

$f_2, x_0 = 0.6$

Table 9. Comparison of different methods

	KT	Sharma	Zheng
$\|x_1 - \alpha\|$	0.798×10^{-3}	0.891×10^{-4}	0.214×10^{-4}
$\|x_2 - \alpha\|$	0.194×10^{-40}	0.541×10^{-45}	0.168×10^{-53}
$\|x_3 - \alpha\|$	0.976×10^{-486}	0.274×10^{-543}	0.386×10^{-642}
COC	11.8387	12.0827	11.9875

$f_3, x_0 = 1.7$

Table 10. Comparison of different methods

	KT	Sharma	Zheng
$\|x_1 - \alpha\|$	0.241×10^{-8}	0.757×10^{-8}	0.221×10^{-7}
$\|x_2 - \alpha\|$	0.137×10^{-99}	0.267×10^{-96}	0.140×10^{-90}
$\|x_3 - \alpha\|$	0.283×10^{-1196}	0.561×10^{-1158}	0.546×10^{-1089}
COC	12.0190	12.0028	12.0001

$f_4, x_0 = 0.35$

Table 11. Comparison of different methods

	KT	Sharma	Zheng
$\|x_1 - \alpha\|$	0.832×10^{-5}	0.412×10^{-7}	0.143×10^{-6}
$\|x_2 - \alpha\|$	0.514×10^{-46}	0.342×10^{-68}	0.156×10^{-62}
$\|x_3 - \alpha\|$	0.231×10^{-542}	0.135×10^{-815}	0.567×10^{-740}
COC	12.9997	12.9997	12.9997

$f_5, x_0 = 1.86$

Table 12. Comparison of different methods

	KT	Sharma	Zheng
$\|x_1 - \alpha\|$	0.871×10^{-9}	0.898×10^{-9}	0.234×10^{-8}
$\|x_2 - \alpha\|$	0.542×10^{-99}	0.675×10^{-98}	0.156×10^{-91}
$\|x_3 - \alpha\|$	0.413×10^{-1198}	0.546×10^{-1160}	0.342×10^{-1090}
COC	12.9997	12.9997	12.9997

In the above tables, error of 1st three iterations are placed along with computational order of convergence of the new method and other methods of same order. It has observed that in 3rd iteration of each method for each nonlinear function, our new method has minimum error as compared to other techniques. The new method has an advantage that it requires less functional evaluations as compared to other ones. Hence, computational cost is reduced and a remarkable efficiency is achieved.

4 Conclusion

We have presented a high order numerical scheme with memory used to solve nonlinear equations. The scheme has convergence order of 12.16 with a remarkable efficiency index 1.8673, requiring four functional evaluations at each iteration. Convergence order of the scheme is proved using matrix method of Herzburger. Our presented scheme is less time consuming with higher efficiency index as compared to other iterative schemes [17]-[18]. The main advantage of this high order scheme is that it is totally derivative free. Hence, it is concluded that our method is derivative free, with higher convergence order and a remarkable efficiency index and is less time consuming.

Competing Interests

Authors have declared that no competing interests exist.

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