The Solutions of the Linear Fractional Diffusion and Diffusion-convection Equations via the Regular Perturbation Method (RPM)

Bationo Jérémie Yiyuréboula [a*] and Bassono Francis [a]

[a]*University Joseph Ki Zerbo, Burkina Faso.*

Authors' contributions

Original Research Article

Abstract

In this paper, we implement Regular Perturbation Method (RPM) for the Solving fractional diffusion and diffusion-convection equations, in order to determine the analytical solutions of some linear fractional diffusion and linear fractional diffusion-convection equations. In general, the solving using this method allow to obtain exact or approximate solutions. For the case of the diffusion and diffusion-convection equations solved in this document, the solutions obtained are exact. By comparing these solutions with those obtained by other researchers using other methods for a certain value of the parameter α, we obtain the same results.

1 Introduction

Today, the notion of fractional calculus is essential in the resolution of partial differential equations. Indeed, several researchers have applied it to several methods such as the method of Adomian, the Homotopy Perturbation method (HPM), etc. However its application remains partial and fragmented. In this document, we will attempt to apply it using the Regular Perturbations method for solving linear fractional diffusion and linear fractional diffusion-convection equations. The objective of this work is to find solutions to fractional equations using the

---

*Corresponding author: E-mail: jeremiebationo@yahoo.fr;

Regular Perturbations method for $0 < \alpha \leq 1$. Secondly, we will compare the solutions obtained with the solutions resulting from another resolution method in a previous search for a given value of α. α is defined in the problem below. Searchers such as Y. MINOUGOU have already solved the case where $\alpha = 1$ with several methods [1].

Let be the following fractional equation (P):

$$(P) \begin{cases} \dfrac{\partial^\alpha u\,(x,t)}{\partial t^\alpha} = f_\varepsilon(t, u(x,t)) & \text{where } 0 < \alpha \leq 1; \\ u(x,0) = g(x) \end{cases}$$

$(x;t) \in \Omega = \mathbb{R} \times [0, +\infty[$ and $u\,(x,t) \in L^2\,(\Omega)$

where g is any function dependent only on x, and f_ε is a continuous function. f_ε can be the right-hand member of a fractional diffusion- convection equation for example, and ε the perturbation coefficient.

2 Preliminaries

In this part, we will recall some very important notions that come into play imperatively in fractional calculus. These are the notions of gamma functions, Beta and Mitag Leffler function as well as some notions of convergence and of solution uniqueness.

2.1 Gamma, Beta and Mittag Leffler functions

2.1.1 Definition 1 [2]

The Gamma function is a function on $]0;1[$, defined by the following integral:

$\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt; \quad s \in \mathbb{C}$ and $Re(s) > 0$.

Thus: $\Gamma(1) = 1$.

$\Gamma(s)$ is a monotonous and strictly decreasing function for $0 \leq s \leq 1$.

2.1.2 Property 1:

$$\bullet \forall s > 0, \ \Gamma(s+1) = s\Gamma(s); \ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$
$$\bullet \forall n \in \mathbb{N}, \ \Gamma(n+1) = n!; \ \text{avec } 0! = 1.$$

2.1.3 Definition 2 [2,3]

The Beta function is the function defined by:

$$\beta(u,v) = \int_0^1 (1-s)^{u-1} s^{v-1} ds;$$

where $Re(u) > 0$ and $Re(v) > 0$.

2.1.4 Property 2:

$\forall u \in \mathbb{C}$; $\forall v \in \mathbb{C}$ where $Re(u) > 0$ and $Re(v) > 0$,

Thus:

$$\beta(u,v) = \beta(v,u) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

2.1.5 Definition 3 [4]

For $s \in \mathbb{C}$, the Mittag-Leffler function denoted $E_\alpha(s)$, with $\alpha > 0$, is defined by:

$$E_\alpha(s) = \sum_{k=0}^{+\infty} \frac{s^k}{\Gamma(k\alpha + 1)}$$

when it depends on a single parameter α.

2.1.6 Property 3:

$\forall s \in \mathbb{C}$ where $Re(s) > 0$, we have:

$E_1(s) = e^s, E_2(s) = ch(\sqrt{s})$, where ch denotes the hyberbolic cosine.

2.1.7 Definition 4 [2]

Let considere $f \in L^1([0,T]), T > 0$. The Riemann-Liouville fractional integral of the function f of order $\alpha \in \mathbb{C}$, $(Re(\alpha) > 0)$ noted I_α is defined by:

$$I_\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau) d\tau$$

$t > 0; x \in \mathbb{R}$.

2.1.8 Definition 5 [5,6,2]

Let considere $f \in L^1([0,T]), T > 0$ a integrable function on $[0,T]$. The fractional derivative in the sens of Riemann-Liouville of the function f of order $\alpha \in \mathbb{C}, (Re(\alpha) > 0)$ noted $D_\alpha f$ is defined by:

$$D_\alpha f(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(x,\tau) d\tau; t > 0; x \in \mathbb{R}.$$

2.2 Concept of convergence and uniqueness of the solution of the problem (P)

Let's consider the problem (P) defined by:

$$(P) \begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = f_\varepsilon(t, u(x,t)) & \text{where } 0 < \alpha \le 1 \text{ and } 0 < \varepsilon \ll 1 \\ u(x,0) = g(x) \end{cases}$$

with its solution in the form:

$$u(x;t) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t)$$

where $g(x) \in C(\Omega)$. we obtain:

$$\sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,\tau)) d\tau$$

As f is linear, we obtain:

$$\sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,0) + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{+\infty} \varepsilon^n \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_n(x,\tau)) d\tau.$$

2.2.1   Proposition 1 [7,8,9,10]

Let's suppose (P) is a linear diffusion equation, where $u(x,t) \in C^2(\Omega)$
and f is linear in u, with $\Omega = \mathbb{R} \times [0;t]$ and $0 \le \tau \le t < T < +\infty$.

If $\exists M = \sup\limits_{x,\tau \in \Omega} \left| \dfrac{\partial^2 u(x,\tau)}{\partial x^2} \right|$ and $m = \sup\limits_{x \in \Omega} |u(x,0)|$ such as: $\forall x, \tau \in \Omega$,
$\left| \dfrac{\partial^2 u(x,\tau)}{\partial x^2} \right| \le M$, and $|u(x,0)| \le m$, then the series $\left(\sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right)$ is absolutely convergent.

2.2.2   Proof 1

We obtain progressively:

$$
\begin{cases}
\varepsilon^0: & |u_0(x,t)| \le |g(x)| \le m \\[2mm]
\varepsilon^1: & |u_1(x,t)| = \left| I_\alpha \left(\dfrac{\partial^2 u_0(x,\tau)}{\partial x^2} \right) \right| \le \left(M \dfrac{T^\alpha}{\Gamma(\alpha+1)} \right) \\[2mm]
... & ... \\[2mm]
\varepsilon^n: & |u_n(x,t)| = \left| I_\alpha \left(\dfrac{\partial^2 u_{n-1}(x,\tau)}{\partial x^2} \right) \right| \le \left(M \dfrac{T^\alpha}{\Gamma(\alpha+1)} \right), n \ge 1
\end{cases}
$$

We have: $\left| \sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right| \le m + \sum\limits_{n=1}^{+\infty} \varepsilon^n \left(M \dfrac{T^\alpha}{\Gamma(\alpha+1)} \right)$

$\qquad \le m + \left(M \dfrac{T^\alpha}{\Gamma(\alpha+1)} \right) \sum\limits_{n=1}^{+\infty} \varepsilon^n = m + \left(M \dfrac{T^\alpha}{\Gamma(\alpha+1)} \right) \left(-1 + \sum\limits_{n=0}^{+\infty} \varepsilon^n \right)$

$\Rightarrow \left| \sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right| \le m + \left(M \dfrac{T^\alpha}{\Gamma(\alpha+1)} \right) \left(-1 + \dfrac{1}{1-\varepsilon} \right)$, because $0 < \varepsilon \ll 1$

As a result, the series $\left(\sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right)$ is absolutely convergent, therefore convergente.

2.2.3   Proposition 2 [7,8,9,10]

Let's suppose (P) a linear diffusion-convection equation, where $u(x,t) \in C(\Omega)$ and $f = \dfrac{\partial^2 u}{\partial x^2} + \lambda \dfrac{\partial u}{\partial x} \in C^2(\Omega)$, with $\Omega = \mathbb{R} \times [0;t]$ and $0 \le \tau \le t < T < +\infty$. If $\exists M = \sup\limits_{x,\tau \in \Omega} \left| \dfrac{\partial^2 u(x,\tau)}{\partial x^2} \right|$, $N = \sup\limits_{x,\tau \in \Omega} \left| \dfrac{\partial u(x,\tau)}{\partial x} \right|$ and $m = \sup\limits_{x \in \Omega} |u(x,0)|$ such as: $\forall x, \tau \in \Omega, \left| \dfrac{\partial^2 u(x,\tau)}{\partial x^2} \right| \le M$, $\left| \dfrac{\partial u(x,\tau)}{\partial x} \right| \le N$ and $|u(x,0)| \le m$, then the series $\left(\sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right)$ is absolutely convergent.

2.2.4   Proof 2

We obtain progressively:

$$
\begin{cases}
\varepsilon^0: & |u_0(x,t)| = |u(x,0)| + \left| \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\lambda \dfrac{\partial u_0(x,\tau)}{\partial x} \right) d\tau \right| \\[3mm]
\varepsilon^1: & |u_1(x,t)| = \left| \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\dfrac{\partial^2 u_0(x,\tau)}{\partial x^2} + \lambda \dfrac{\partial u_1(x,\tau)}{\partial x} \right) d\tau \right| \\[3mm]
... & ... \\[3mm]
\varepsilon^n: & |u_n(x,t)| = \left| \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\dfrac{\partial^2 u_{n-1}(x,\tau)}{\partial x^2} + \lambda \dfrac{\partial u_n(x,\tau)}{\partial x} \right) d\tau \right|, n \ge 1
\end{cases}
$$

$$\implies \begin{cases} \varepsilon^0: & |u_0(x,t)| \leq m + \left(\lambda N \dfrac{T^\alpha}{\Gamma(\alpha+1)}\right) \\[2mm] \varepsilon^1: & |u_1(x,t)| \leq \left((M+\lambda N)\dfrac{T^\alpha}{\Gamma(\alpha+1)}\right) \\[2mm] \dots & \dots \\[2mm] \varepsilon^n: & |u_n(x,t)| \leq \left((M+\lambda N)\dfrac{T^\alpha}{\Gamma(\alpha+1)}\right), n \geq 1 \end{cases}$$

We have: $\left|\sum\limits_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right| \leq m + \sum\limits_{n=0}^{+\infty}\varepsilon^n\left((M+\lambda N)\dfrac{T^\alpha}{\Gamma(\alpha+1)}\right)$

$$\leq m + \left((M+\lambda N)\dfrac{T^\alpha}{\Gamma(\alpha+1)}\right)\sum\limits_{n=0}^{+\infty}\varepsilon^n$$

$$\left|\sum\limits_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right| \leq m + \dfrac{\left((M+\lambda N)\dfrac{T^\alpha}{\Gamma(\alpha+1)}\right)}{1-\varepsilon}, \text{ because } 0 < \varepsilon \ll 1$$

As a result, the series $\left(\sum\limits_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right)$ is absolutely convergent, therefore convergent.

2.2.5 Proposition 3 [8]

The equation (P_1) and (P_2) each admit a unique solution which is written in the form:

$u(x,t) = u(x,0) + I_\alpha\left(f(\tau,u(x,\tau))\right),$ where $I_\alpha\left(f\right)$ denotes the fractional integral of f defined by:
$I_\alpha\left(f(\tau,u(x,\tau))\right) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,u_0(x,\tau))d\tau$

2.2.6 Proof 3

Now let's show the uniqueness of the solution.

Let's suppose there are two distinct solutions $u(x,t)$ and $v(x,t)$ such that:
there is $w(x,t) = u(x,t) - v(x,t) \neq 0$
Considering the following algorithms of u and v :

$$\begin{cases} u_0(x,t) = u(x,0) + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,u_0(x,\tau))d\tau \\[2mm] u_1(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,u_1(x,\tau))d\tau \\[2mm] \dots \\[2mm] u_n(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,u_n(x,\tau))d\tau \end{cases}$$

and

$$\begin{cases} v_0(x,t) = v(x,0) + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,v_0(x,\tau))d\tau \\[2mm] v_1(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,v_1(x,\tau))d\tau \\[2mm] \dots \\[2mm] v_n(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,v_n(x,\tau))d\tau \end{cases}$$

We have:

$$
\begin{cases}
w_0(x,t) = w(x,0) + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, w_0(x,\tau)) d\tau = 0 \\[2mm]
w_1(x,t) = \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, w_1(x,\tau)) d\tau = 0 \\[2mm]
... \\[2mm]
w_n(x,t) = \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, w_n(x,\tau)) d\tau = 0
\end{cases}
$$

Because from the conditions to the limits, it happens that

$$u(x,0) = g(x) = v(x,0) \Rightarrow w(x,0) = 0$$

In addition, $f(\tau, w_0(x,\tau))$ is a linear function of $w(x,0)$

Where from $w_0(x,0) = 0 \Rightarrow f(\tau, w_0(x,\tau)) = 0 \Rightarrow w_0(x,t) = 0$;absurd.

Similary,$f(\tau, w_n(x,\tau))$ is a linear function of $g(x), n \geq 1$

It follows that $w_n(x,t) = 0 = u_n(x,t) - v_n(x,t),\ \forall n \geq 0$.

Thus $u(x,t) = v(x,t)$ hence the uniqueness of the solution.

3 Applications

3.1 Fractional diffusion linear equation [11,12,13,14]

Let's considere the following fonctional equation to fractional order:

$Lu = f + Ru + Nu$ and let (P_1) be the problem to be studied defined by:

$$
(P_1) \begin{cases}
\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} & \text{where } 0 < \alpha \leq 1 \text{ and } 0 < \varepsilon \ll 1; \\[2mm]
u(x,0) = \sin \omega x
\end{cases}
$$

$(x;t) \in \Omega = \mathbb{R} \times [0, +\infty[$ and $u(x,t) \in L^2(\Omega)$

Let's find the solution of (P_1) in the form of:

$u(x,t) = \sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t)$, where $0 < \varepsilon \ll 1$

As

$$L(u(x,t)) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$$

and

$$Ru(x;t) = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$L(u(x,t)) = Ru(x;t)$$

with

$$L^{-1}(u(x,t)) = I_\alpha(u(x,t))$$

By applying L^{-1} in equation (P_1), we obtain:

$$u(x,t) - u(x,0) = L^{-1}Ru(x,t)$$

which is equivalent to:

$$u_n(x,t) = u(x,0) + I_\alpha \left(\varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} \right)$$

$$\Longrightarrow \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = u(x,0) + \sum_{n=0}^{+\infty} I_\alpha \left(\varepsilon^{n+1} \frac{\partial^2 u_n(x,t)}{\partial x^2} \right)$$

The following equations are obtained successively by identification of powers of ε:

$$\begin{cases} \varepsilon^0: & u_0(x,t) = u(x,0) = g(x) = \sin \omega x \\ \varepsilon^1: & u_1(x,t) = I_\alpha \left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2} \right) \\ ... & ... \\ \varepsilon^n: & u_n(x,t) = I_\alpha \left(\dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2} \right); n \geq 1 \end{cases}$$

This leads us to solve the following equations:

$$\begin{cases} \varepsilon^0: \begin{cases} \dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0 \\ u_0(x,0) = \sin(\omega x) \end{cases} \\ \varepsilon^1: \begin{cases} \dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_0(x,t)}{\partial x^2} \\ u_1(x,0) = 0 \end{cases} \\ ... \\ \varepsilon^n: \begin{cases} \dfrac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2}, \forall n \geq 1 \\ u_n(x,0) = 0, n \geq 1 \end{cases} \end{cases} \qquad (1.2)$$

Let's solve the system ε^0 :

$$\frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0 \text{ with } u_0(x,0) = \sin(\omega x)$$

In Caputo's sense, the solution to this equation is in the form:

$$u_0(x;t) - u_0(x,0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_0(x,\tau)) d\tau$$

As $f(\tau, u_0(\tau)) = 0$, then,

$$u_0(x;t) = u_0(x;0)$$

where from $u_0(x;t) = u_0(x;0) = \sin(\omega x)$

The solution to this equation is:

$$u_0(x,t) = \sin(\omega x)$$

Let's solve the system ε^1 :

$$\frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_0(x,t)}{\partial x^2} \text{ with } u_1(x,0) = 0$$

In Caputo's sense, the solution to this equation is in the form:

$$u_1(x;t) - u_1(x;0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_1(x,\tau)) d\tau$$

As $f(\tau, u_1(\tau)) = \dfrac{\partial^2 u_0(x,\tau)}{\partial x^2}$,

$$\Rightarrow u_1(x;t) = \frac{-\omega^2 \sin(\omega x)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau$$

Let's posing $\tau = tz$, $d\tau = tdz$; When $\tau \to t$, $alors\ z \to 1$

We obtain: $u_1(x;t) = \dfrac{-\omega^2 \sin(\omega x)}{\Gamma(\alpha)} \int_0^1 (t-tz)^{\alpha-1} tdz$

$$\Rightarrow u_1(x;t) = \frac{-\omega^2 \sin(\omega x)}{\Gamma(\alpha)} t^\alpha \int_0^1 (1-z)^{\alpha-1} z^{1-1} dz$$

$$\Rightarrow u_1(x;t) = \frac{-\omega^2 \sin(\omega x)}{\Gamma(\alpha+1)} t^\alpha$$

The solution to this equation is:

$$u_1(x;t) = \sin(\omega x) \frac{(-\omega^2 t^\alpha)}{\Gamma(\alpha+1)}$$

Let's solve the system ε^2 :

$$\frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_1(x,t)}{\partial x^2} \text{ with } u_2(x,0) = 0$$

Using the same approach of resolution as in ε^0 and ε^1, we obtain as solution of the system ε^2 :

$$u_2(x;t) = \sin(\omega x) \frac{(-\omega^2 t^\alpha)^2}{\Gamma(2\alpha+1)}$$

In summary, we have:

$$\begin{cases} u_0(x,t) = \sin(\omega x) \\ u_1(x,t) = \sin(\omega x) \dfrac{(-\omega^2 t^\alpha)}{\Gamma(1\alpha+1)} \\ u_2(x,t) = \sin(\omega x) \dfrac{(-\omega^2 t^\alpha)^2}{\Gamma(2\alpha+1)} \\ ... \\ u_n(x,t) = \sin(\omega x) \dfrac{(-\omega^2 t^\alpha)^n}{\Gamma(n\alpha+1)} \end{cases} \tag{1.3}$$

By posing, $\quad \varphi_n(x,t) = \sum\limits_{k=0}^n \varepsilon^k u_k(x,t)$ \hfill (1.4)

The solution of this equation (P_1) is:

$$u(x,t) = \lim_{n \to +\infty} \varphi_n(x,t) = \sin(\omega x) \lim_{n \to +\infty} \sum_{k=0}^n \frac{(-\varepsilon\omega^2 t^\alpha)^k}{\Gamma(k\alpha+1)} = \sin(\omega x) \sum_{k=0}^{+\infty} \frac{(-\varepsilon\omega^2 t^\alpha)^k}{\Gamma(k\alpha+1)}$$

We thus obtain: $\quad u((x,t) = \sin(\omega x) E_\alpha(-\varepsilon\omega^2 t^\alpha)$ \hfill (1.5)

Note: When $\alpha = 1$, ,the solution of system is:$u(x,t) = \sin(\omega x) e^{-\varepsilon\omega^2 t}$.

Thus, this solution is the same as that obtain by Y. Minougou in his thesis [1]where he used the Adomian method with $\alpha = 1$.

3.2 Fractional diffusion-convection linear equation [11,12,13,14]

Let (P_2) be the problem to be studied defined by:

$$(P_2): \begin{cases} \dfrac{\partial^\alpha u\,(x,t)}{\partial t^\alpha} = \varepsilon\dfrac{\partial^2 u\,(x,t)}{\partial x^2} + \lambda\dfrac{\partial u\,(x,t)}{\partial x} \text{where } 0 < \varepsilon \ll 1; \lambda > 0 \text{ and } 0 < \alpha \le 1 \\ u\,(x;0) = \sin(\omega x) \end{cases}$$

$(x;t) \in \Omega = \mathbb{R} \times [0, +\infty[$ and $u\,(x,t) \in L^2\,(\Omega)$

Let's posing $L(u(x,t)) = \dfrac{\partial^\alpha u\,(x,t)}{\partial t^\alpha}$ and $Ru\,(x;t) = \varepsilon\dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda\dfrac{\partial u(x,t)}{\partial x}$

$\implies L^{-1}(u(x,t)) = I_\alpha(u(x,t))$ where $L(u(x,t)) = Ru\,(x;t)$

By L^{-1} to the equation (P_2), we obtain: $u(x,t) - u(x,0) = L^{-1}Ru(x,t)$

Finding the solution in the form of: $u(x,t) = \sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t)$ $\hspace{3cm}$ (2.1)

Which equals: $\sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = u(x,0) + \sum\limits_{n=0}^{+\infty} \varepsilon^n I_\alpha R\,(u_n(x,t))$

or $\hspace{1.5cm} \sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = u(x,0) + \sum\limits_{n=0}^{+\infty} \varepsilon^n I_\alpha\left(\varepsilon\dfrac{\partial^2 u_n(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_n(x,t)}{\partial x}\right)$

We successively obtain the following equations:

$$\begin{cases} \varepsilon^0: & u_0(x,t) = u(x,0) + \lambda I_\alpha\left(\dfrac{\partial u_0(x,t)}{\partial x}\right) \\ \varepsilon^1: & u_1(x,t) = I_\alpha\left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_1(x,t)}{\partial x}\right) \\ \varepsilon^2: & u_2(x,t) = I_\alpha\left(\dfrac{\partial^2 u_1(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_2(x,t)}{\partial x}\right) \\ ... & ... \\ \varepsilon^n: & u_n(x,t) = I_\alpha\left(\dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_n(x,t)}{\partial x}\right); n \ge 1 \end{cases}$$

This leads us to solve the following equations:

$$\varepsilon^0: \begin{cases} \dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = \lambda\dfrac{\partial u_0(x,t)}{\partial x} \\ u_0(x,0) = \sin(\omega x) \end{cases}$$

$$\varepsilon^1: \begin{cases} \dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_1(x,t)}{\partial x} \\ u_1(x,0) = 0 \end{cases}$$

$$\varepsilon^2: \begin{cases} \dfrac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_1(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_2(x,t)}{\partial x} \\ u_2(x,0) = 0 \end{cases} \hspace{2cm} (2.2)$$

$$\hspace{0.3cm}...$$

$$\varepsilon^n: \begin{cases} \dfrac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_n(x,t)}{\partial x}; n > 0 \\ u_n(x,0) = 0; n > 0 \end{cases}$$

Let's solve the system ε^0 :

$$\dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = \lambda\dfrac{\partial u_0(x,t)}{\partial x} \quad \text{with } u_0(x,0) = \sin(\omega x)$$

In Caputo's sense, the solution to this equation is in the form:

$$u_0(x;t) - u_0(x,0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_0(x,\tau)) d\tau$$

As $f(\tau, u_0(x,\tau)) = \lambda \frac{\partial u_0(x,\tau)}{\partial x}$,then,

$$u_0(x;t) = u_0(x,0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \lambda \frac{\partial u_0(x,\tau)}{\partial x}) d\tau$$

We have : $u_0(x,t) = \sum_{k=0}^{+\infty} u_{0_k}(x,t)$

Thus :
$$\begin{cases} u_{0_0}(x;t) = u_0(x;0) = \sin(\omega x) \\ u_{0_{n+1}}(x;t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial u_{0_n}(x,\tau)}{\partial x} d\tau \; ; \; n \geq 0 \end{cases}$$

we obtain after calculation:

$$\begin{cases} u_{0_{2p}}(x;t) = \sin(\omega x) (-1)^p \frac{(\lambda \omega t^\alpha)^{2p}}{\Gamma(2p\alpha+1)}; p \geq 0 \\ u_{0_{2p+1}}(x;t) = \cos(\omega x)(-1)^p \frac{(\lambda \omega t^\alpha)^{2p+1}}{\Gamma((2p+1)\alpha+1)}; p \geq 0 \end{cases}$$

We have: $u_0(x,t) = \sum_{k=0}^{+\infty} u_{0_k}(x,t) = \sum_{k=0}^{+\infty} u_{0_{2k}}(x,t) + \sum_{k=0}^{+\infty} u_{0_{2k+1}}(x,t)$

The solution to this equation is:

$$u_0(x,t) = \sin(\omega x) \sum_{k=0}^{+\infty} (-1)^k \frac{(\lambda \omega t^\alpha)^{2k}}{\Gamma(2k\alpha+1)} + \cos(\omega x) \sum_{k=0}^{+\infty} (-1)^k \frac{(\lambda \omega t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^1 :

We have: $\frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda \frac{\partial u_1(x,t)}{\partial x}$, with $u_1(x,0) = 0$

In Caputo's sense, the solution to this equation is in the form:

$$u_1(x;t) = u_1(x,0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_1(x,\tau)) d\tau$$

As $f(\tau, u_1(\tau)) = \frac{\partial^2 u_0(x,\tau)}{\partial x^2} + \lambda \frac{\partial u_0(x,\tau)}{\partial x}$,then,

$$u_1(x;t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (\frac{\partial^2 u_0(x,\tau)}{\partial x^2} + \lambda \frac{\partial u_1(x,\tau)}{\partial x}) d\tau$$

$$u_1(x,t) = \sum_{k=0}^{+\infty} u_{1_k}(x,t)$$

Thus:
$$\begin{cases} u_{1_0}(x;t) = u_1(x;0) = 0 \\ u_{1_{n+1}}(x;t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (\frac{\partial^2 u_{0_n}(x,\tau)}{\partial x^2} + \lambda \frac{\partial u_{1_n}(x,\tau)}{\partial x}) d\tau \end{cases}$$

Using the values of u_{2_k} obtained in ε^2,we obtain after calculation:

$$\begin{cases} u_{1_{2k}}(x;t) = (-1)^k 2k \frac{(\lambda \omega t^\alpha)^{2k-1}}{\Gamma(2k\alpha+1)} (\omega^2 t^\alpha) \cos(\omega x); k \geq 0 \\ u_{1_{2k+1}}(x;t) = -(-1)^k (2k+1) \frac{(\lambda \omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)} (-\omega^2 t^\alpha) \sin(\omega x); k \geq 0 \end{cases}$$

We have: $u_1(x,t) = \sum_{k=0}^{+\infty} u_{1_{2k}}(x,t) + \sum_{k=0}^{+\infty} u_{1_{2k+1}}(x,t)$

The solution to this equation is:

$$u_1(x,t) = (\omega^2 t^\alpha)\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2k\,\frac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma(2k\alpha+1)}$$
$$-(\omega^2 t^\alpha)\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+1)\,\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^2 :

We have: $\dfrac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_1(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_2(x,t)}{\partial x}$, with $u_2(x,0)=0$

Using the same approach of resolution as in ε^0 and ε^1 and using the values of u_{1_k} obtained in ε^1, we obtain after calculation:

$$\begin{cases} u_{2_{2k}}(x,t) = -\dfrac{1}{2}(-1)^k 2k(2k-1)(\omega^4 t^{2\alpha})\sin(\omega x)\dfrac{(\lambda\omega t^\alpha)^{2k-2}}{\Gamma((2k)\alpha+1)}; k\geq 0 \\[3mm] u_{2_{2k+1}}(x,t) = -\dfrac{1}{2}(-1)^k(2k+1)(2k)(\omega^4 t^{2\alpha})\cos(\omega x)\dfrac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma((2k+1)\alpha+1)}; k\geq 0 \end{cases}$$

We have: $u_2(x,t) = \sum\limits_{k=0}^{+\infty} u_{2_{2k}}(x,t) + \sum\limits_{k=0}^{+\infty} u_{2_{2k+1}}(x,t)$

The solution to this equation is:

$$u_2(x,t) = -\frac{1}{2}(\omega^2 t^\alpha)^2\sin(\omega x)\sum_{k=1}^{+\infty}(-1)^k 2k(2k-1)\frac{(\lambda\omega t^\alpha)^{2k-2}}{\Gamma((2k)\alpha+1)}$$
$$-\frac{1}{2}(\omega^2 t^\alpha)^2\cos(\omega x)\sum_{k=1}^{+\infty}(-1)^k(2k+1)(2k)\frac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^3 :

We have: $\dfrac{\partial^\alpha u_3(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_2(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_3(x,t)}{\partial x}$, with $u_3(x,0)=0$

Using the same approach of resolution as in ε^0 and ε^1 and using the values of u_{2_k} obtained in ε^2, we obtain after calculation:

$$\begin{cases} u_{3_{2k}}(x,t) = -(-1)^k\dfrac{1}{3!}2k(2k-1)(2k-2)(\omega^6 t^{3\alpha})\cos(\omega x)\dfrac{(\lambda\omega\,t^\alpha)^{2k-3}}{\Gamma(2k\alpha+1)}; k\geq 0 \\[3mm] u_{3_{2k+1}}(x,t) = (-1)^k\dfrac{1}{3!}(2k+1)(2k)(2k-1)(\omega^6 t^{3\alpha})\sin(\omega x)\dfrac{(\lambda\omega\,t^\alpha)^{2k-2}}{\Gamma((2k+1)\alpha+1)}; k\geq 0 \end{cases}$$

We have : $u_3(x,t) = \sum\limits_{k=0}^{+\infty} u_{3_{2k}}(x,t) + \sum\limits_{k=0}^{+\infty} u_{3_{2k+1}}(x,t)$

The solution to this equation is:

$$u_3(x,t) = -\frac{1}{3!}(\omega^2 t^\alpha)^3\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2k(2k-1)(2k-2)\frac{(\lambda\omega\,t^\alpha)^{2k-3}}{\Gamma(2k\alpha+1)}$$
$$+\frac{1}{3!}(\omega^2 t^\alpha)^3\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+1)(2k)(2k-1)\frac{(\lambda\omega\,t^\alpha)^{2k-2}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^4 :

We have: $\dfrac{\partial^\alpha u_4(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_3(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_4(x,t)}{\partial x}$, with $u_4(x,0)=0$

Using the same approach of resolution as in ε^0 and ε^1 and using the values of u_{3_k} obtained in ε^3, we obtain after calculation:

$$\begin{cases} u_{4_{2k}}(x,t) = (-1)^k \dfrac{1}{4!} 2k(2k-1)(2k-2)(2k-3)(\omega^8 t^{4\alpha}) \sin(\omega x) \dfrac{(\lambda\omega t^\alpha)^{2k-4}}{\Gamma(2k\alpha+1)}; k \geq 0 \\[2ex] u_{4_{2k+1}}(x,t) = (-1)^k \dfrac{1}{4!} (2k+1)(2k)(2k-1)(2k-2)(\omega^8 t^{4\alpha}) \cos(\omega x) \dfrac{(\lambda\omega t^\alpha)^{2k-3}}{\Gamma((2k+1)\alpha+1)}; k \geq 0 \end{cases}$$

We have: $\quad u_4(x,t) = \sum\limits_{k=0}^{+\infty} u_{4_{2k}}(x,t) + \sum\limits_{k=0}^{+\infty} u_{4_{2k+1}}(x,t)$

The solution of this equation is:

$$u_4(x,t) = \frac{(\omega^2 t^\alpha)^4}{4!} \sin(\omega x) \sum_{k=0}^{+\infty} (-1)^k 2k(2k-1)(2k-2)(2k-3) \frac{(\lambda\omega t^\alpha)^{2k-4}}{\Gamma(2k\alpha+1)}$$
$$+ \frac{(\omega^2 t^\alpha)^4}{4!} \cos(\omega x) \sum_{k=0}^{+\infty} (-1)^k (2k+1)(2k)(2k-1)(2k-2) \frac{(\lambda\omega t^\alpha)^{2k-3}}{\Gamma((2k+1)\alpha+1)}$$

From close to close, we obtain:

$$\begin{cases} u_0(x,t) = \sin(\omega x) \sum\limits_{k=0}^{+\infty} (-1)^k \dfrac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2k\alpha+1)} \\[2ex] \qquad\quad + \cos(\omega x) \sum\limits_{k=0}^{+\infty} (-1)^k \dfrac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)} \\[2ex] u_1(x,t) = (\omega^2 t^\alpha) \cos(\omega x) \sum\limits_{k=1}^{+\infty} (-1)^k 2k \dfrac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma(2k\alpha+1)} \\[2ex] \qquad\quad - (\omega^2 t^\alpha) \sin(\omega x) \sum\limits_{k=0}^{+\infty} (-1)^k (2k+1) \dfrac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)} \\[2ex] u_2(x,t) = \dfrac{(\omega^2 t^\alpha)^2}{2!} \sin(\omega x) \sum\limits_{k=1}^{+\infty} (-1)^k 2k(2k-1) \dfrac{(\lambda\omega\, t^\alpha)^{2k-2}}{\Gamma(2k\alpha+1)} \\[2ex] \qquad\quad - \dfrac{(\omega^2 t^\alpha)^2}{2!} \cos(\omega x) \sum\limits_{k=1}^{+\infty} (-1)^k (2k+1)(2k) \dfrac{(\lambda\omega\, t^\alpha)^{2k-1}}{\Gamma((2k+1)\alpha+1)} \\[2ex] u_3(x,t) = -\dfrac{(\omega^2 t^\alpha)^3}{3!} \cos(\omega x) \sum\limits_{k=2}^{+\infty} (-1)^k 2k(2k-1)(2k-2) \dfrac{(\lambda\omega\, t^\alpha)^{2k-3}}{\Gamma(2k\alpha+1)} \\[2ex] \qquad\quad + \dfrac{(\omega^2 t^\alpha)^3}{3!} \sin(\omega x) \sum\limits_{k=1}^{+\infty} (-1)^k (2k+1)(2k)(2k-1) \dfrac{(\lambda\omega\, t^\alpha)^{2k-2}}{\Gamma((2k+1)\alpha+1)} \\[2ex] u_4(x,t) = \dfrac{(\omega^2 t^\alpha)^4}{4!} \sin(\omega x) \sum\limits_{k=2}^{+\infty} (-1)^k 2k(2k-1)(2k-2)(2k-3) \dfrac{(\lambda\omega t^\alpha)^{2k-4}}{\Gamma(2k\alpha+1)} \\[2ex] \qquad\quad + \dfrac{(\omega^2 t^\alpha)^4}{4!} \cos(\omega x) \sum\limits_{k=2}^{+\infty} (-1)^k (2k+1)(2k)(2k-1)(2k-2) \dfrac{(\lambda\omega t^\alpha)^{2k-3}}{\Gamma((2k+1)\alpha+1)} \\[2ex] \dots \end{cases} \quad : \quad (2.3)$$

Let's posing:

$$\varphi_n(x,t) = \sum_{k=0}^{n} \varepsilon^k u_k(x,t) \qquad\qquad (2.4)$$

with

$$u(x,t) = \lim_{n \to +\infty} \varphi_n(x,t)$$

The solution of this equation (P_2) is:

$$u(x,t) = \varepsilon^0 u_0(x,t) + \varepsilon^1 u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + \varepsilon^4 u_4(x,t) + \dots$$

Which allows us to get by reducing the smallest value from k to 0, we obtain:

$$
\left\{
\begin{aligned}
u(x,t) = &[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\,\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2k\alpha+1)} + \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\,\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)}]+ \\
&(-\varepsilon\omega^2 t^\alpha)[\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+1)\,\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma(2(k+1)\alpha+1)}+ \\
&\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+1)\,\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)}]+ \\
&\frac{(-\varepsilon\omega^2 t^\alpha)^2}{2!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+1)(2(k+1)-1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2(k+1)\alpha+1)}+ \\
&\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2(k+1)+1)(2(k+1))\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2(k+1)+1)\alpha+1)}]+ \\
&\frac{(-\varepsilon\omega^2 t^\alpha)^3}{3!}[\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+2)(2(k+2)-1)(2(k+2)-2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma(2(k+2)\alpha+1)}+ \\
&\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2(k+1)+1)(2(k+1))(2(k+1)-1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2(k+1)+1)\alpha+1)}]+ \\
&\frac{(-\varepsilon\omega^2 t^\alpha)^4}{4!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+2)(2(k+2)-1)(2(k+2)-2)(2(k+2)-3)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2(k+2)\alpha+1)}+ \\
&\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2(k+2)+1)(2(k+2))(2(k+2)-1)(2(k+2)-2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2(k+2)+1)\alpha+1)}]+...
\end{aligned}
\right.
$$

Which still gives:

$$
\left\{
\begin{aligned}
u(x,t) = &[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\,\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2k\alpha+1)} + \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\,\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)}]+ \\
&(-\varepsilon\omega^2 t^\alpha)[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+1)\,\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)}+ \\
&\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+2)\,\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+2)\alpha+1)}]+ \\
&\frac{(-\varepsilon\omega^2 t^\alpha)^2}{2!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+2)(2k+1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+2)\alpha+1)}+ \\
&\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+3)(2k+2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+3)\alpha+1)}]+ \\
&\frac{(-\varepsilon\omega^2 t^\alpha)^3}{3!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+3)(2k+2)(2k+1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+3)\alpha+1)}+ \\
&\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+4)(2k+3)(2k+2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+4)\alpha+1)}]+ \\
&\frac{(-\varepsilon\omega^2 t^\alpha)^4}{4!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+4)(2k+3)(2k+2)(2k+1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+4)\alpha+1)} \\
&\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+5)(2k+4)(2k+3)(2k+2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+5)\alpha+1)}]+...
\end{aligned}
\right.
$$

In the end, we obtain:

$$
\begin{aligned}
u(x,t) = \sum_{p=0}^{+\infty}\frac{(-\varepsilon\omega^2 t^\alpha)^p}{p!}[&\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k A_{2k+p}^p\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+p)\alpha+1)} \\
&+ \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k A_{2k+p+1}^p\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+p+1)\alpha+1)}]
\end{aligned}
\tag{2.5}
$$

where A_m^r avec $m \geq r$ is the arrangement of r in m

Note: For $\alpha = 1$, we have:

$$\sum_{k=0}^{+\infty} (-1)^k A_{2k+p}^p \frac{(\lambda \omega t^\alpha)^{2k}}{\Gamma((2k+p)\alpha + 1)} = \sum_{k=0}^{+\infty} (-1)^k \frac{(\lambda \omega t)^{2k}}{(2k)!} = \cos(\lambda \omega t)$$

$$\text{et } \sum_{k=0}^{+\infty} (-1)^k A_{2k+p+1}^p \frac{(\lambda \omega t^\alpha)^{2k+1}}{\Gamma((2k+p+1)\alpha + 1)} = \sum_{k=0}^{+\infty} (-1)^k \frac{(\lambda \omega t)^{2k+1}}{(2k+1)!} = \sin(\lambda \omega t)$$

$$\implies u(x,t) = [\sin(\omega x) \cos(\lambda \omega t) + \cos(\omega x) \sin(\lambda \omega t)] \sum_{p=0}^{+\infty} \frac{(-\varepsilon \omega^2 t)^p}{p!}$$

$$u(x,t) = \sin(\omega x + \lambda \omega t) e^{-\varepsilon \omega^2 t}$$

Thus, this solution is the same as that obtain by Y. Minougou in his thesis [1] where he used the Adomian method with $\alpha = 1$.

4 Conclusion

The resolution of fractional equations by the Perturbation method regular is very tedious because it requires much more vigilance in calculations. The use of the Mittag-Leffler function is essential. In the event that $\alpha = 1$, the solutions become simpler because they appeal to the exponential function. In short, the method, although tedious, is very effective because it provides solutions to most linear partial differential equations. However, it requires mastery of the basic notions of numerical series. In addition, we have in perspective to apply it to the nonlinear equations.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Minougou Y. Contribution des méthodes SBA et Laplace-SBA à la résolution des modèles de convection-diffusion-réaction : Thèse de doctorat Université Joseph KI-ZERBO, soutenu le 10-06-2019; 2019.

[2] Ali K. Résolution des équations aux dérivées partielles linéaires et non-linéaires moyennant des approches analytiques.Extension aux cas d'EDP d'ordre fractionnaire.Thèse de doctorat en science, mathématiques appliquée, Université Ferhat A; 2019.

[3] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. Wiley, New York; 1993.

[4] Simon T. Fonctions de Mittag-Leffler et processus de Levy Stables sans sauts négatifs, Laboratoire Paul Painlev e, U. F. R. de Math ematiques, Universit e de Lille 1, F-59655 Villeneuve d'Ascq Cedex. Adresse electronique: simon@math.univ-lille1.fr; 2000.

[5] Mbaiguesse D, Jeremie B, Abbo B, Pare Y. UniversityJoseph KI-ZERBO ;University of Njamena Tchad**Analytical solution of some nonlinear fractional integrodifferential equations of the Fedholm second kind by a new approximation technique of the numerical SBA method, KAMATE Adama; 2022.

[6] Mbaiguesse D, Jeremie B, Abbo B, Pare Y. UniversityJoseph KI-ZERBO ;University of Njamena Tchad**Analytical solution of some nonlinear fractional integrodifferential equations of the Fedholm second kind by a new approximation technique of the numerical SBA method, KAMATE Adama; 2022.

[7] Some B. Convergence of the Adomian method applied to Friedholm integro-differential equations. Africa Mat. 2001;3(14):71-88.

[8] Ouedraogo S, Abbo B, Yaro R, Pare Y. Convergence of a new approch of Adomian's method for solving volterra fractional integral of second kind , UFR/SEA, Departement Mathematiques et Informatique (Burkina Faso); 2020.

[9] Pare Y. Resolution de quelques equations fonctionnelles par la numerique SBA (SOME BLAISE-ABBO), Thèse de Doctorat unique. Université de Ouagadougou, mai 2010, UFR/SEA, Departement Mathematiques et Informatique (Burkina Faso); 2010.

[10] Pare Y, Bassono F, Some B. A new technique for numerical resolution of few non linear integral equations of Fredholm by SBA method. Far East Journal of Applied Mathematics. 2012;70(1):21-33. BBAS SETIF 1.

[11] Abbo B. Nouvel algorithme numérique de résolution des Equations Ordinaires (EDO) et des Equations aux dérivées partielles (EDP) non linéaires.Thèse de Doctorat unique. Université de Ouagadougou, UFR/SEA, Département Mathématiques et Informatique (Burkina Faso); 2007.

[12] Bassono F. Résolution numérique comparée de quelques équations fonctionnelles par la méthode SBA, décompositionnelle d'Adomia et Perturbations, thèse de doctorat unique, Université de Ouagadougou, janvier 2013, UFR SEA; 2013.

[13] Abbaoui K, Cherruault Y. The decomposion method applies to the Cauchy problem. Kybernetes. 1999;28(1):68-74 .

[14] Podlubny I. Fractional differential equations. Academic, New York; 1999.

_____

_____